Differntiability

1 Mark Questions

1. Write the derivative of $\sin x$ with respect to $\cos x$. **Delhi 2014C**

Let $u = \sin x$

On differentiating u w.r.t. x, we get

$$\frac{du}{dx} = \cos x \quad \text{...(i)}$$

and $v = \cos x$

On differentiating v w.r.t. x, we get

$$\frac{dv}{dx} = -\sin x \quad \text{...(ii)}$$

Now, $\frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv}$

$= -\frac{\cos x}{\sin x}$ [from Eqs. (i) and (ii)]

$\Rightarrow \frac{du}{dv} = -\cot x$ **(1)**

2. If $\cos y = x \cos(a + y)$, where $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$. **Foreign 2014**

Given, $\cos y = x \cos(a+y)$

$$\Rightarrow \quad x = \frac{\cos y}{\cos(a+y)}$$

On differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\cos(a+y)\frac{d}{dy}(\cos y) - \cos y \frac{d}{dy}[\cos(a+y)]}{\cos^2(a+y)}$$

[by using quotient rule]

$$\Rightarrow \frac{dx}{dy} = \frac{\cos(a+y)(-\sin y) - \cos y[-\sin(a+y)]}{\cos^2(a+y)}$$

$$= \frac{\cos y \sin(a+y) - \cos(a+y)\sin y}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$[\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin a}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a} \quad \textbf{(1)}$$

Hence proved.

3. If $y = \sin^{-1}\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\}$ and $0 < x < 1$, then find $\frac{dy}{dx}$.

All India 2014C; Delhi 2010

Firstly, convert the given expression in $\sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ form and then put $x = \sin\phi$ and $y = \sin\theta$. Now, simplify the resulting expression and differentiate it.

Given, $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$

Above equation can be rewritten as

Above equation can be rewritten as

$$y = \sin^{-1}[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}]$$

$$[\because\ x = (\sqrt{x})^2]$$

Now, put $\sqrt{x} = \sin\theta$ and $x = \sin\phi$, so that

$$\theta = \sin^{-1}\sqrt{x} \text{ and } \phi = \sin^{-1}x, \text{ we get}$$

$$y = \sin^{-1}[\sin\phi\sqrt{1-\sin^2\theta} - \sin\theta\sqrt{1-\sin^2\phi}]$$

$$\Rightarrow \quad y = \sin^{-1}[\sin\phi\cos\theta - \sin\theta\cos\phi]$$

$$[\because\ \sqrt{1-\sin^2 x} = \cos x]$$

$$\Rightarrow \quad y = \sin^{-1}\sin(\phi-\theta)$$

$$[\because\ \sin\phi\cos\theta - \sin\theta\cos\phi = \sin(\phi-\theta)]$$

$$\Rightarrow \quad y = \phi - \theta \qquad [\because \sin^{-1}\sin\theta = \theta]$$

$$\Rightarrow \quad y = \sin^{-1}x - \sin^{-1}\sqrt{x}$$

$$[\because\ \phi = \sin^{-1}x \text{ and } \theta = \sin^{-1}\sqrt{x}]$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{d}{dx}(\sqrt{x})$$

$$\left[\because \frac{d}{d\theta}(\sin^{-1}\theta) = \frac{1}{\sqrt{1-\theta^2}}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\left[\because \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}\right]$$

Hence, $$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}} \qquad \textbf{(1)}$$

Alternate Method

Use the formula,

$$\sin^{-1}x - \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$$

and then differentiate with respect to x to get and then differentiate with respect to x to get the required value.

$\because \quad y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$

$\Rightarrow \quad y = \sin^{-1}[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}]$

$[\because \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\} = \sin^{-1}x - \sin^{-1}y]$

Here, $\quad x = x$ and $y = \sqrt{x}$

$\therefore \quad y = \sin^{-1}x - \sin^{-1}\sqrt{x}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x) - \frac{d}{dx}(\sin^{-1}\sqrt{x})$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}} \qquad \textbf{(1)}$$

4. If $e^x + e^y = e^{x+y}$, prove that $\frac{dy}{dx} + e^{y-x} = 0$.

Foregin 2014

Given, $e^x + e^y = e^{x+y}$ ...(i)

On dividing Eq. (i) by e^{x+y}, we get

$$e^{-y} + e^{-x} = 1 \qquad \text{...(ii)}$$

On differentiating both sides of Eq. (ii) w.r.t. x, we get

$$e^{-y} \cdot \left(\frac{-dy}{dx}\right) + e^{-x}(-1) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-e^{-x}}{e^{-y}} = -e^{(y-x)}$$

$$\Rightarrow \quad \frac{dy}{dx} + e^{(y-x)} = 0 \qquad \textbf{(1)}$$

5. Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if

$x = ae^{\theta}(\sin\theta - \cos\theta)$ and $y = ae^{\theta}(\sin\theta + \cos\theta)$.

All India 2014

Given, $x = ae^{\theta}(\sin\theta - \cos\theta)$

On differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}[e^{\theta}\sin\theta - e^{\theta}\cos\theta]$$

$$= a\left[\frac{d}{d\theta}(e^{\theta}\sin\theta) - \frac{d}{d\theta}(e^{\theta}\cos\theta)\right]$$

$$= a\left[e^{\theta}\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(e^{\theta}) - e^{\theta}\frac{d}{d\theta}(\cos\theta) - \cos\theta\frac{d}{d\theta}(e^{\theta})\right]$$

$$= a[e^{\theta}\cos\theta + e^{\theta}\sin\theta - e^{\theta}(-\sin\theta) - e^{\theta}\cos\theta]$$

$$= a[e^{\theta}\cos\theta + e^{\theta}\sin\theta + e^{\theta}\sin\theta - e^{\theta}\cos\theta]$$

$$\Rightarrow \frac{dx}{d\theta} = a[2e^{\theta}\sin\theta] = 2ae^{\theta}\sin\theta \quad \ldots(i)$$

Also, $y = ae^{\theta}(\sin\theta + \cos\theta)$

On differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(e^{\theta}\sin\theta) + \frac{d}{d\theta}(e^{\theta}\cos\theta)\right]$$

$$= a\left[e^{\theta}\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(e^{\theta}) + e^{\theta}\frac{d}{d\theta}(\cos\theta) + \cos\theta\frac{d}{d\theta}(e^{\theta})\right]$$

$$= a[e^{\theta}\cos\theta + e^{\theta}\sin\theta - e^{\theta}\sin\theta + e^{\theta}\cos\theta]$$

$$= a[2e^{\theta}\cos\theta]$$

$$\Rightarrow \quad \frac{dy}{d\theta} = 2ae^{\theta}\cos\theta \qquad \ldots\text{(ii)}$$

Now, $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta}$

[from Eqs. (i) and (ii)]

$$= \cot\theta$$

At $\theta = \frac{\pi}{4}, \frac{dy}{dx} = \cot\frac{\pi}{4}$

Hence, $\frac{dy}{dx} = 1$ $\left[\because \cot\frac{\pi}{4} = 1\right]$ **(1)**

6. If $x = a\left(\cos t + \log\tan\frac{t}{2}\right), y = a\sin t$, then evaluate $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$. **Delhi 2014C**

Given, $x = a\left(\cos t + \log \tan\frac{t}{2}\right)$

and $y = a \sin t$. ...(i)

Now, $x = a\left(\cos t + \log \tan\frac{t}{2}\right)$

On differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = a\left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\log \tan\frac{t}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$\left[\because \frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \sec^2\frac{t}{2}\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}} \times \sec^2\frac{t}{2} \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{\frac{\sin t/2}{\cos t/2}} \times \frac{1}{\cos^2\frac{t}{2}} \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin\frac{t}{2} \cdot \cos\frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] \quad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= a\left(\frac{1-\sin^2 t}{\sin t}\right) \Rightarrow \frac{dx}{dt} = a\left(\frac{\cos^2 t}{\sin t}\right) \quad \text{...(ii)}$$

[illegible]

$$[\because 1 - \sin^2\theta = \cos^2\theta]$$

and $\qquad y = a \sin t$

On differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = a \cos t \qquad \text{...(iii)}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)}$$

[from Eqs. (ii) and (iii)]

$$= \frac{a \cos t}{a \cos^2 t} \times \sin t = \tan t$$

On differentiating both sides of above equation w.r.t.x, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t)$$

$$= \frac{d}{dt}(\tan t)\frac{dt}{dx} \quad \left[\because \frac{d}{dx} f(t) = \frac{d}{dt} f(t) \cdot \frac{dt}{dx}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \sec^2 t \times \frac{\sin t}{a \cos^2 t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\sin t \sec^4 t}{a}$$

Now, on putting $t = \frac{\pi}{3}$, we get

$$\left[\frac{d^2y}{dx^2}\right]_{t=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3} \times \sec^4 \frac{\pi}{3}}{a} = \frac{\frac{\sqrt{3}}{2} \times (2)^4}{a}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\frac{\sqrt{3}}{2} \times 16}{a}$$

[illegible]

Hence, $\frac{d^2y}{dx^2} = \frac{8\sqrt{3}}{a}$ **(1)**

7. If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

Foreign 2014

Given, $x^m y^n = (x+y)^{m+n}$

On taking log both sides, we get

$$\log(x^m y^n) = \log(x+y)^{m+n}$$

$$\Rightarrow \quad \log(x^m) + \log(y^n) = (m+n)\log(x+y)$$

$$\Rightarrow \quad m\log x + n\log y = (m+n)\log(x+y)$$

On differentiating w.r.t. x, we get

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{m}{x} - \frac{(m+n)}{x+y} = \left(\frac{m+n}{x+y} - \frac{n}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow \left[\frac{my+ny-nx-ny}{y(x+y)}\right]\frac{dy}{dx} = \frac{mx+my-mx-nx}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{my-nx}{y}\right] = \frac{my-nx}{x}$$

Hence, $\frac{dy}{dx} = \frac{y}{x}$ **(1)**

8. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$,

show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.

Foreign 2014

Given, $x = a\cos\theta + b\sin\theta$ ...(i)

and $y = a\sin\theta - b\cos\theta$ ...(ii)

On differentiating Eq. (i) w.r.t. θ, we get

$$\frac{dx}{d\theta} = -a\sin\theta + b\cos\theta$$

On differentiating Eq. (ii) w.r.t. θ, we get

$$\frac{dy}{d\theta} = a\cos\theta + b\sin\theta$$

Now, $$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a\cos\theta + b\sin\theta}{-a\sin\theta + b\cos\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{-y} \quad \begin{bmatrix} \because x = a\cos\theta + b\sin\theta, \\ y = a\sin\theta - b\cos\theta \end{bmatrix}$$

$$\Rightarrow y\frac{dy}{dx} = -x \Rightarrow y\frac{dy}{dx} + x = 0 \quad \text{...(iii)}$$

On differentiating Eq. (iii) w.r.t. x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$$

$$\Rightarrow y\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{-x}{y}\right) + 1 = 0 \quad \left[\because \frac{dy}{dx} = \frac{-x}{y}\right]$$

$$\Rightarrow y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \quad \textbf{(1)}$$

Hence proved.

9. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ w.r.t. $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$. **Delhi 2014**

Let $u = \tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

Then, $u = \tan^{-1}\left[\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right]$

$= \tan^{-1}\left[\dfrac{\sqrt{\sin^2\theta}}{\cos\theta}\right] = \tan^{-1}\left[\dfrac{\sin\theta}{\cos\theta}\right]$

$= \tan^{-1}[\tan\theta] = \theta$

$\Rightarrow \quad u = \cos^{-1} x \qquad [\because x = \cos\theta]$

On differentiating w.r.t. x, we get

$$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Again, let $v = \cos^{-1}(2x\sqrt{1-x^2})$

Put $\quad x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

Then, $\quad v = \cos^{-1}[2\cos\theta\sqrt{1-\cos^2\theta}]$

$= \cos^{-1}[2\cos\theta\sin\theta]$

$\left[\begin{array}{l}\because \cos^2\theta + \sin^2\theta = 1 \\ \sin\theta = \sqrt{1-\cos^2\theta}\end{array}\right]$

$= \cos^{-1}[\sin 2\theta]$

$= \cos^{-1}\left[\cos\left(\dfrac{\pi}{2} - 2\theta\right)\right] = \dfrac{\pi}{2} - 2\theta$

$\Rightarrow \quad v = \dfrac{\pi}{2} - 2\cos^{-1} x$

On differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Now, $\quad \dfrac{du}{dv} = \dfrac{du}{dx} \times \dfrac{dx}{dv}$

$= -\dfrac{1}{\sqrt{1-x^2}} \times \dfrac{\sqrt{1-x^2}}{2} = -\dfrac{1}{2} \qquad$ **(1)**

10. If $y = x^x$, then prove that

$$\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$$

Delhi 2014

Given, $y = x^x$

On taking log both sides, we get

$$\log y = \log x^x$$

$$\Rightarrow \quad \log y = x \log x$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \log x$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (1+\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1+\log x) \quad \text{...(i)}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = y\frac{d}{dx}(1+\log x) + (1+\log x)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = y \times \frac{1}{x} + (1+\log x)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x} + (1+\log x)\frac{dy}{dx} \quad \text{...(ii)}$$

Now, we have to prove that

$$\frac{d^2y}{dx^2} - \frac{-1}{y}\left(\frac{dy}{dx}\right)^2 \frac{-y}{x} = 0$$

$$\text{LHS} = \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x}$$

$$= \frac{y}{x} + (1+\log x)\frac{dy}{dx} - \frac{1}{y}[y(1+\log x)]^2 - \frac{y}{x}$$

[from Eqs. (i) and (ii)]

$$= \frac{y}{x} + (1+\log x)\,y(1+\log x)$$

$$- \frac{1}{y}[y^2(1+\log x)^2] - \frac{y}{x} \quad \text{[from Eq. (i)]}$$

$$= y(1+\log x)^2 - y(1+\log x)^2 = 0 = \text{RHS}$$

Hence proved. (1)

11. Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ w.r.t. $\sin^{-1}(2x\sqrt{1-x^2})$. **Delhi 2014**

Let $u = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$, then

$$u = \tan^{-1}\left[\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right]$$

$$\Rightarrow \quad u = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right] \left[\begin{array}{l}\because \sin^2\theta + \cos^2\theta = 1 \\ \Rightarrow \cos\theta = \sqrt{(1-\sin^2\theta)}\end{array}\right]$$

$$\Rightarrow \quad u = \tan^{-1}(\tan\theta) \Rightarrow u = \theta \quad \Rightarrow u = \sin^{-1} x$$

On differentiating w.r.t. x, we get

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \text{...(i)}$$

Again, let $v = \sin^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$, then

$$v = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$

$$\Rightarrow \quad v = \sin^{-1}(2\sin\theta\cos\theta)$$

$$\Rightarrow \quad v = \sin^{-1}(\sin 2\theta) \Rightarrow v = 2\theta$$

$$\Rightarrow \quad v = 2\sin^{-1} x$$

On differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}} \qquad \text{...(ii)}$$

Now, $$\frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv}$$

$$\Rightarrow \quad \frac{du}{dv} = \frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{2}$$

$$\therefore \quad \frac{du}{dv} = \frac{1}{2}$$ **(1)**

12. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, when $x \neq 0$.

Delhi 2014

Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \cot\theta \Rightarrow \theta = \cot^{-1} x$, then

$$u = \tan^{-1}\left[\frac{\sqrt{1+\cot^2\theta}-1}{\cot\theta}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{\operatorname{cosec}^2\theta}-1}{\cot\theta}\right]$$

$$= \tan^{-1}\left[\frac{\operatorname{cosec}\theta - 1}{\cot\theta}\right] = \tan^{-1}\left[\frac{1-\sin\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}\right]$$

$[\because \sin^2 x + \cos^2 x = 1]$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)}\right]$$

$[\because a^2 - b^2 = (a+b)(a-b)]$

$$= \tan^{-1}\left[\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}\right] = \tan^{-1}\left[\frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}\right]$$

[∵ dividing numerator and denominator by $\cos\theta/2$]

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$

$$\Rightarrow \quad u = \frac{\pi}{4} - \frac{\theta}{2} = \frac{\pi}{4} - \frac{\cot^{-1} x}{2}$$

On differentiating w.r.t. x, we get

$$\frac{du}{dx} = \frac{1}{2(1 + x^2)} \quad \text{...(i)}$$

$$\left[\because \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1 + x^2}\right]$$

Again, let $\quad v = \sin^{-1}\left(\frac{2x}{1 + x^2}\right)$

Put $\quad x = \tan\theta \Rightarrow \theta = \tan^{-1} x$, then we get

$$v = \sin^{-1}\left[\frac{2\tan\theta}{1 + \tan^2\theta}\right]$$

$$\Rightarrow v = \sin^{-1}[\sin 2\theta] \Rightarrow v = 2\theta \Rightarrow v = 2\tan^{-1} x$$

On differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \frac{2}{1 + x^2} \quad \text{...(ii)}$$

Now, $\quad \frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv} = \frac{1}{2(1 + x^2)} \times \frac{(1 + x^2)}{2}$

[from Eqs. (i) and (ii)]

$$\Rightarrow \quad \frac{du}{dv} = \frac{1}{4} \quad \textbf{(1)}$$

13. If $y = Pe^{ax} + Qe^{bx}$, then show that

$$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0.$$

All India 2014, 2009C

Given, $y = Pe^{ax} + Qe^{bx}$ ...(i)

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = P\frac{d}{dx}(e^{ax}) + Q\frac{d}{dx}(e^{bx})$$

$$\Rightarrow \frac{dy}{dx} = Pa\,e^{ax} + Qb\,e^{bx} \quad ...(ii)$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = Pa\frac{d}{dx}(e^{ax}) + bQ\frac{d}{dx}(e^{bx})$$

$$= Pa(a\,e^{ax}) + bQ(b\,e^{bx})$$

$$= a^2P\,e^{ax} + b^2Q\,e^{bx} \quad ...(iii)$$

Now, LHS $= \frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby$

On putting values from Eqs. (i), (ii) and (iii), we get

LHS $= a^2P\,e^{ax} + b^2Q\,e^{bx}$

$- (a+b)(aP\,e^{ax} + bQ\,e^{bx}) + ab(P\,e^{ax} + Q\,e^{bx})$

$= a^2P\,e^{ax} + b^2Q\,e^{bx} - a^2P\,e^{ax} - ab\,Q\,e^{bx}$

$\quad - ab\,P\,e^{ax} - b^2Q\,e^{bx} + ab\,P\,e^{ax} + ab\,Q\,e^{bx}$

$= 0 =$ RHS **(1)**

Hence proved.

14. If $x = \cos t\,(3 - 2\cos^2 t)$ and $y = \sin t\,(3 - 2\sin^2 t)$, then find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.

All India 2014

Given, $x = \cos t\,(3 - 2\cos^2 t)$

$\Rightarrow \quad x = 3\cos t - 2\cos^3 t$

On differentiating w.r.t. t, we get

$$\frac{dx}{dt} = 3(-\sin t) - 2(3)\cos^2 t\,(-\sin t)$$

$$\Rightarrow \quad \frac{dx}{dt} = -3\sin t + 6\cos^2 t\sin t \qquad \text{...(i)}$$

Also, $y = \sin t\,(3 - 2\sin^2 t)$

$\Rightarrow \quad y = 3\sin t - 2\sin^3 t$

On differentiating w.r.t. t, we get

$$\frac{dy}{dt} = 3\cos t - 2\times 3\times \sin^2 t\cos t$$

$$\Rightarrow \quad \frac{dy}{dt} = 3\cos t - 6\sin^2 t\cos t \qquad \text{...(ii)}$$

Now, $$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \frac{3\cos t - 6\cos t\sin^2 t}{-3\sin t + 6\cos^2 t\sin t}$$

[from Eqs. (i) and (ii)]

$$= \frac{\cos t - 2\cos t\sin^2 t}{-\sin t + 2\cos^2 t\sin t}$$

At $t = \frac{\pi}{4}$,

$$\left[\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \frac{\cos\frac{\pi}{4} - 2\cos\frac{\pi}{4}\sin^2\frac{\pi}{4}}{-\sin\frac{\pi}{4} + 2\cos^2\frac{\pi}{4}\sin\frac{\pi}{4}}$$

$$= \frac{\frac{1}{\sqrt{2}} - 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^2}{-\frac{1}{\sqrt{2}} + 2\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}\right)} = \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = 0$$

15. If $(x-y)\,e^{\frac{x}{x-y}}=a$. Prove that $y\frac{dy}{dx}+x=2y$.

Delhi 2014C

Given, $$(x-y)\cdot e^{\frac{x}{x-y}}=a$$

On taking log both sides, we get

$$\log\left[(x-y)\cdot e^{\frac{x}{x-y}}\right]=\log a$$

$$\Rightarrow \quad \log(x-y)+\log e^{\frac{x}{x-y}}=\log a$$

$[\because \log(mn)=\log m+\log n]$

$$\Rightarrow \quad \log(x-y)+\frac{x}{x-y}\log_e e=\log a$$

$$\Rightarrow \quad \log(x-y)+\frac{x}{x-y}=\log a$$

$[\because \log_e=1]$

On differentiating w.r.t. x, we get

$$\frac{d}{dx}[\log(x-y)]+\frac{d}{dx}\left(\frac{x}{x-y}\right)=\frac{d}{dx}(\log a)$$

$$\Rightarrow \frac{1}{x-y}\frac{d}{dx}(x-y)+\frac{(x-y)\frac{d}{dx}(x)-x\frac{d}{dx}(x-y)}{(x-y)^2}=0$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right)=\frac{u\frac{d}{dx}(v)-v\frac{d}{dx}(u)}{v^2}\right]$$

$$\Rightarrow \quad \frac{1}{x-y}\cdot(1-y')+\frac{(x-y)-x(1-y')}{(x-y)^2}=0$$

where, $y'=dy/dx$

$\Rightarrow \quad (x-y)(1-y')+x-y-x(1-y')=0$

$\Rightarrow \quad yy'+x-2y=0$

$$\Rightarrow \quad y\frac{dy}{dx}+x=2y \qquad \textbf{(1)}$$

Hence proved.

16. If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t\cos t)$, then find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. **Delhi 2014C**

Given, $x = a(\cos t + t \sin t)$

On differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = a(-\sin t + 1 \cdot \sin t + t \cos t)$$

$$\Rightarrow \quad \frac{dx}{dt} = a\, t \cos t$$

and $\quad y = a(\sin t - t \cos t)$

On differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = a(\cos t - \cos t \cdot 1 + t \sin t) = at \sin t \quad \text{...(i)}$$

Now, $\quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{at \sin t}{at \cos t} = \tan t$

Again differentiating both sides w.r.t. x, we get

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(\tan t)\frac{dt}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{1}{\frac{dx}{dt}} = \frac{\sec^2 t}{at \cos t} = \frac{\sec^3 t}{at}$$

At $t = \frac{\pi}{4}$,

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{t=\frac{\pi}{4}} = \frac{\sec^3\left(\frac{\pi}{4}\right)}{a\left(\frac{\pi}{4}\right)} = \frac{(\sqrt{2})^3 \cdot 4}{2\pi}$$

$$= \frac{8\sqrt{2}}{a\pi} \quad \textbf{(1)}$$

17. If $y = \tan^{-1}\left(\frac{a}{x}\right) + \log\sqrt{\frac{x-a}{x+a}}$, prove that $\frac{dy}{dx} = \frac{2a^3}{x^4 - a^4}$.

All India 2014C

Given, $y = \tan^{-1}\left(\frac{a}{x}\right) + \log\sqrt{\frac{x-a}{x+a}}$

$$= \tan^{-1}\left(\frac{a}{x}\right) + \log\left(\frac{x-a}{x+a}\right)^{1/2}$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{a}{x}\right) + \frac{1}{2}[\log(x-a) - \log(x+a)]$$

$$\left[\because \log\frac{m}{n} = \log m - \log n\right]$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+\frac{a^2}{x^2}} \cdot \left(\frac{-a}{x^2}\right) + \frac{1}{2}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

$$\left[\because \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \text{ and } \frac{d}{dx}(\log x) = \frac{1}{x}\right]$$

$$= \frac{-a}{x^2+a^2} + \frac{a}{x^2-a^2}$$

$$= \frac{-x^2a + a^3 + x^2a + a^3}{x^4 - a^4}$$

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a^3}{x^4 - a^4} \qquad \textbf{(1)}$$

Hence proved.

18. If $x = a \sin 2t\,(1 + \cos 2t)$ and $y = b \cos 2t\,(1 - \cos 2t)$, then show that at $t = \frac{\pi}{4}, \left(\frac{dy}{dx}\right) = \frac{b}{a}$. **All India 2014**

Given, $x = a \sin 2t\,(1 + \cos 2t)$

$\Rightarrow \quad x = a \sin 2t\,(2 \cos^2 t)$

$[\because \cos 2\theta = 2\cos^2\theta - 1]$

$\Rightarrow \quad x = 2a \sin 2t \cos^2 t$

On differentiating x w.r.t. t, we get

$$\frac{dx}{dt} = 2a\left[\sin 2t \frac{d}{dt}(\cos^2 t) + \cos^2 t \frac{d}{dt}(\sin 2t)\right]$$

$$= 2a\,[\sin 2t\,\{2 \cos t\,(-\sin t)\} + 2\cos^2 t\,(\cos 2t)]$$

$$= 2a\,[-\sin^2 2t + 2\cos^2 t \cos 2t]$$

$[\because 2 \sin\theta \cos\theta = \sin 2\theta]$

Also, $\quad y = b \cos 2t\,(1 - \cos 2t)$

$= b \cos 2t\,(2 \sin^2 t)$

$[\because \cos 2\theta = 1 - 2\sin^2\theta]$

$= 2b \cos 2t \sin^2 t$

On differentiating y w.r.t. t, we get

$$\frac{dy}{dt} = 2b\left[\cos 2t \frac{d}{dt}(\sin^2 t) + \sin^2 t \frac{d}{dt}(\cos 2t)\right]$$

$$= 2b\,[\cos 2t\,(2 \sin t \cos t) + \sin^2 t\,(-\sin 2t) \cdot 2]$$

$$= 2b\,[\cos 2t \sin 2t - 2\sin^2 t \sin 2t]$$

Now, $\quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2b\,[\cos 2t \sin 2t - 2\sin^2 t \sin 2t]}{2a\,[2\cos^2 t \cos 2t - \sin^2 2t]}$$

At $t = \frac{\pi}{4}$,

$$\left[\cos 2\left(\frac{\pi}{4}\right) \sin 2\left(\frac{\pi}{4}\right) \right.$$

$$\left[\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \frac{b}{a}\frac{\left[-2\sin^2\left(\frac{\pi}{4}\right)\sin 2\left(\frac{\pi}{4}\right)\right]}{\left[2\cos^2\left(\frac{\pi}{4}\right)\cos 2\left(\frac{\pi}{4}\right)-\sin^2 2\left(\frac{\pi}{4}\right)\right]}$$

$$= \frac{b}{a}\frac{\left[\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)-2\times\left(\frac{1}{\sqrt{2}}\right)^2\sin\left(\frac{\pi}{2}\right)\right]}{\left[2\times\left(\frac{1}{\sqrt{2}}\right)^2\cos\left(\frac{\pi}{2}\right)-\sin^2\left(\frac{\pi}{2}\right)\right]}$$

$$\left[\because \sin\frac{\pi}{4}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}\right]$$

$$= \frac{b}{a}\frac{[0\times 1 - 1\times 1]}{[1\times 0 - 1]}$$

$$= \frac{b}{a}\left(\frac{-1}{-1}\right) \qquad \left[\because \sin\frac{\pi}{2}=1, \cos\frac{\pi}{2}=0\right]$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \frac{b}{a} \qquad \textbf{(1)}$$

Hence proved.

19. If $(\tan^{-1} x)^y + y^{\cot x} = 1$, then find dy/dx.

All India 2014C

Let $u = (\tan^{-1} x)^y$ and $v = y^{\cot x}$

Then, given equation becomes $u + v = 1$

On differentiating both sides w.r.t. x, we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \qquad \text{...(i)}$$

Now, $u = (\tan^{-1} x)^y$

On taking log both sides, we get

$$\log u = y \log(\tan^{-1} x)$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = \frac{dy}{dx}\log(\tan^{-1} x) + \frac{y}{\tan^{-1} x\,(1+x^2)}$$

$$\Rightarrow \frac{du}{dx} = (\tan^{-1} x)^y$$

$$\left[\frac{dy}{dx}\log(\tan^{-1}x)+\frac{y}{\tan^{-1}x\,(1+x^2)}\right]\quad\ldots\text{(ii)}$$

Also, $\quad v = y^{\cot x}$

On taking log both sides, we get

$$\log v = \cot x \log y$$

On differentialting w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = -\operatorname{cosec}^2 x \log y + \frac{\cot x}{y}\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} = y^{\cot x}\left[-\operatorname{cosec}^2 x \log y + \frac{\cot x}{y}\frac{dy}{dx}\right]\quad\ldots\text{(iii)}$$

On putting values from Eqs. (ii) and (iii) in Eq. (i), we get

$$(\tan^{-1}x)^y\left[\frac{dv}{dx}\log(\tan^{-1}x)+\frac{y}{\tan^{-1}x\,(1+x^2)}\right]$$

$$+\,y^{\cot x}\left[-\operatorname{cosec}^2 x \log y + \frac{\cot x}{y}\frac{dv}{dx}\right]=0$$

$$\Rightarrow \frac{dy}{dx}\left[(\tan^{-1}x)^y \log(\tan^{-1}x) + \cot x\, y^{\cot x - 1}\right]$$

$$= -\left[\frac{y}{1+x^2}(\tan^{-1}x)^{y-1} - y^{\cot x}\operatorname{cosec}^2 x \log y\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left[\frac{y}{1+x^2}(\tan^{-1}x)^{y-1} - y^{\cot x}\cdot \operatorname{cosec}^2 x \log y\right]}{[(\tan^{-1}x)^y \log(\tan^{-1}x) + \cot x\, y^{\cot x-1}]}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left[\frac{y}{1+x^2}(\tan^{-1}x)^{y-1} - y^{\cot x} \operatorname{cosec}^2 x \log y\right]}{[(\tan^{-1}x)^y \log(\tan^{-1}x) + \cot x\, y^{\cot x-1}]}$$

(1)

4 marks Questions

20. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, then prove that $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$. **Delhi 2013C**

To prove, $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$

Given, $x = 2\cos\theta - \cos 2\theta$

and $y = 2\sin\theta - \sin 2\theta$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta$$

and $\frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$ **(1)**

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2(\cos\theta - \cos 2\theta)}{2(-\sin\theta + \sin 2\theta)} \quad \textbf{(1)}$$

$$= \frac{2\sin\left(\frac{\theta + 2\theta}{2}\right)\sin\left(\frac{2\theta - \theta}{2}\right)}{2\left[\cos\left(\frac{2\theta + \theta}{2}\right)\sin\left(\frac{2\theta - \theta}{2}\right)\right]} \quad \textbf{(1)}$$

$$\left[\begin{array}{l} \because \cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right) \\ \text{and } \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \end{array}\right]$$

$$= \frac{\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)} = \tan\left(\frac{3\theta}{2}\right) = \text{RHS} \quad \textbf{(1)}$$

Hence proved.

21. If $y = (\sin x)^x + \sin^{-1}\sqrt{x}$, then find $\frac{dy}{dx}$.

Delhi 2013C, 2009; All India 2009C

Given, $y = (\sin x)^x + \sin^{-1}\sqrt{x}$ ...(i)

Let $u = (\sin x)^x$ ...(ii)

Then, Eq. (i) becomes, $y = u + \sin^{-1}\sqrt{x}$ ...(iii)

On taking log both sides of Eq. (ii), we get

$$\log u = x \log \sin x \quad \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}(\log \sin x) + \log \sin x \frac{d}{dx}(x)$$

[by product rule]

$$\Rightarrow \frac{du}{dx} = u\left[x \times \frac{1}{\sin x}\frac{d}{dx}(\sin x) + \log \sin x(1)\right] \textbf{(1)}$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\left[\frac{x}{\sin x} \times \cos x + \log \sin x\right]$$

[from Eq. (ii)]

$$\Rightarrow \frac{du}{dx} = (\sin x)^x[x \cot x + \log \sin x] \quad ...(iv)$$

On differentiating both sides of Eq.(iii) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x}) \quad \textbf{(1)}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^x[x \cot x + \log \sin x]$$

$$+ \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} \quad \text{[from Eq. (iv)]} \ \textbf{(1)}$$

22. If $y = x\log\left(\frac{x}{a+bx}\right)$, then prove that

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2.$$

Delhi 2013C

To prove, $x^3 \frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$

Given, $y = x\log\left(\frac{x}{a+bx}\right)$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = x\frac{d}{dx}\log\left(\frac{x}{a+bx}\right) + \log\left(\frac{x}{a+bx}\right)\frac{d}{dx}(x)$$

[by product rule]

$$= x\left(\frac{1}{\frac{x}{a+bx}}\right)\frac{d}{dx}\left(\frac{x}{a+bx}\right) + \log\left(\frac{x}{a+bx}\right)\cdot 1$$

$\left[\because \frac{d}{dx}(\log x) = \frac{1}{x}\right]$

$$= (a+bx)\left[\frac{(a+bx)(1) - x(b)}{(a+bx)^2}\right] + \log\left(\frac{x}{a+bx}\right)$$

[by quotient rule]

$$= (a+bx)\left[\frac{a}{(a+bx)^2}\right] + \log\left(\frac{x}{a+bx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{a}{a+bx} + \log\left(\frac{x}{a+bx}\right)$$ ...(i) **(1)**

Now, $x\frac{dy}{dx} - y = \frac{ax}{a+bx} + x\log\left(\frac{x}{a+bx}\right) - y$

$$= \frac{ax}{a+bx} + x\log\left(\frac{x}{a+bx}\right) - x\log\left(\frac{x}{a+bx}\right)$$

$$\Rightarrow x\frac{dy}{dx} - y = \frac{ax}{a+bx}$$ ...(ii) **(1)**

From Eq. (i), we have

$$\frac{dy}{dx} = \frac{a}{a+bx} + \log\left(\frac{x}{a+bx}\right)$$

On differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{-a}{(a+bx)^2} \cdot b + \frac{1}{\left(\frac{x}{a+bx}\right)} \cdot \left\{\frac{(a+bx)\cdot 1 - x \cdot b}{(a+bx)^2}\right\}$$

$$= \frac{-ab}{(a+bx)^2} + \frac{a+bx}{x}\left\{\frac{a+bx-bx}{(a+bx)^2}\right\}$$

$$= \frac{-ab}{(a+bx)^2} + \frac{a+bx}{x} \times \frac{a}{(a+bx)^2}$$

$$= \frac{-ab}{(a+bx)^2} + \frac{a}{x(a+bx)} \qquad \textbf{(1)}$$

$$\Rightarrow x^3 \frac{d^2y}{dx^2} = -\frac{abx^3}{(a+bx)^2} + \frac{ax^2}{(a+bx)}$$

[multiplying both sides by x^3]

$$= \frac{ax^2}{(a+bx)^2}\{-bx + (a+bx)\}$$

$$= \frac{a^2x^2}{(a+bx)^2}$$

$$\Rightarrow x^3 \frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx}\right)^2 = \left(x\frac{dy}{dx} - y\right)^2$$

[from Eq. (ii)] **(1)**

Hence proved.

23. Differentiate the following function with respect to x.

$(\log x)^x + x^{\log x}$ **Delhi 2013**

Let $y=(\log x)^x+x^{\log x}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\{(\log x)^x+x^{\log x}\}$$

$$=\frac{d}{dx}(\log x)^x+\frac{d}{dx}(x^{\log x})$$

$$=(\log x)^x\frac{d}{dx}[\{x\log(\log x)\}]$$

$$+x^{\log x}\frac{d}{dx}(\log x\log x)$$

$$\left[\because \frac{d}{dx}(u^v)=u^v\frac{d}{dx}(v\log u)\right] \quad \textbf{(2)}$$

$$=(\log x)^x\left\{x\left(\frac{1}{\log x}\right)\frac{1}{x}+\log(\log x)\right\}$$

$$+x^{\log x}\left[2(\log x)\frac{1}{x}\right]$$

$$\left[\because \frac{d}{dx}\log(\log x)=\frac{1}{\log x}\times\frac{1}{x},\right.$$

$$\left.\frac{d}{dx}(\log x\log x)=\frac{d}{dx}\{(\log x)^2\}=2(\log x)\frac{1}{x}\right]$$

$$=(\log x)^x\left\{\frac{1}{\log x}+\log(\log x)\right\}$$

$$+2\left(\frac{\log x}{x}\right)x^{\log x} \quad \textbf{(2)}$$

24. If $y=\log[x+\sqrt{x^2+a^2}]$, then show that

$$(x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0.$$ **Delhi 2013**

Given, $y = \log[x + \sqrt{x^2 + a^2}]$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}} \frac{d}{dx}\left(x + \sqrt{x^2 + a^2}\right)$$

$$\left[\because \frac{d}{dx}(\log x) = \frac{1}{x}\frac{d}{dx}(x)\right] \textbf{(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}}\left(1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right)$$

$$\left[\because \frac{d}{dx}(\sqrt{x^2 + a^2}) = \frac{1 \times 2x}{2\sqrt{x^2 + a^2}}\right]$$

$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left(\frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}}\left(\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}(\sqrt{x^2 + a^2}) = 1$$ **(1)**

Again on differentiating both sides w.r.t. to x, we get

$$\sqrt{x^2 + a^2}\,\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(\sqrt{x^2 + a^2}) = \frac{d(1)}{dx}$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2}(\sqrt{x^2+a^2}) + \frac{1 \cdot 2x \cdot \frac{dy}{dx}}{2\sqrt{x^2+a^2}} = 0 \qquad \textbf{(1)}$$

$$\left[\because \frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}\right]$$

$$\Rightarrow \qquad (x^2 + a^2) + \frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0 \qquad \textbf{(1)}$$

Hence proved.

25. Show that the function $f(x) = |x-3|, x \in R$, is continuous but not differentiable at $x = 3$.

Delhi 2013

Firstly, to check the differentiability of the function $f(x)$ at $x=3$. Find LHD and RHD, if LHD $\neq$ RHD, then function is not differentiable and then we check continuity of the function at $x=3$ by showing LHL = RHL = $f(3)$.

Given, $f(x)=|x-3|$

First, we check the differentiability of the given function $f(x)$ at $x=3$.

$$\text{LHD} = f'(3^-) = \lim_{h\to 0} \frac{f(3-h)-f(3)}{-h}$$

$$\left[\because Lf'(a) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}\right]$$

$$= \lim_{h\to 0} \frac{|3-h-3|-|3-3|}{-h}$$

$$= \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1 \qquad [\because |-x| = x] \textbf{ (1)}$$

$$\text{RHD} = f'(3^+) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$

$$\left[\because Rf'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right]$$

$$= \lim_{h\to 0} \frac{|3+h-3|-|3-3|}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1 \qquad [\because |-x| = x] \textbf{ (1)}$$

Since, LHD $\neq$ RHD at $x = 3$

So, f is not differentiable.

Now, we check the continuity of the given function $f(x)$ at $x=3$

$$\therefore \quad \text{LHL} = \lim_{x \to 3^-} |x-3|$$

$$= \lim_{h \to 0} |3-h-3| \quad [\text{put } x = 3-h]$$

$$= \lim_{h \to 0} |-h| = 0 \quad \textbf{(1)}$$

$$\text{RHL} = \lim_{x \to 3^+} |x-3|$$

$$= \lim_{h \to 0} |3+h-3| = \lim_{h \to 0} |h| = 0 \quad [\text{put } x = x+h]$$

and $f(3) = |3-3| = 0$

Thus, $\text{LHL} = \text{RHL} = f(3)$

Hence, f is continuous at $x=3$. **(1)**

26. If $x=a\sin t$ and $y=a(\cos t+\log\tan(t/2))$, then find $\dfrac{d^2y}{dx^2}$.

Delhi 2013

Given, $y = a\left(\cos t + \log \tan \frac{t}{2}\right)$ and $x = a \sin t$

On differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = a\left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log \tan \frac{t}{2}\right)\right]$$

$$\Rightarrow \frac{dy}{dt} = a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \times \frac{d}{dt}\left(\tan \frac{t}{2}\right)\right]$$

$$\Rightarrow \frac{dy}{dt} = a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \times \sec^2\left(\frac{t}{2}\right) \times \frac{1}{2}\right] \quad \textbf{(1)}$$

$$= a\left[-\sin t + \frac{1}{2} \times \frac{\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)} \times \frac{1}{\cos^2\left(\frac{t}{2}\right)}\right]$$

$$= a\left[-\sin t + \frac{1}{2 \sin \frac{t}{2} . \cos \frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] \qquad \left[\because \sin t = 2 \sin \frac{t}{2} . \cos \frac{t}{2}\right]$$

$$= a\left[\frac{1 - \sin^2 t}{\sin t}\right] = a \frac{\cos^2 t}{\sin t}$$

$[\because \sin^2 t + \cos^2 t = 1 \Rightarrow 1 - \sin^2 t = \cos^2 t]$ **(1)**

and $\frac{dx}{dt} = \frac{d}{dt}(a \sin t) = a \cos t$ **(1/2)**

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left[\frac{a \cos^2 t}{\sin t}\right]}{a \cos t} = \cot t$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}(\cot t)$ **(1/2)**

$= \frac{d}{dt}(\cot t) \times \left(\frac{dt}{dx}\right) = (-\operatorname{cosec}^2 t)\left(\frac{dt}{dx}\right)$

$= -(\operatorname{cosec}^2 t) . \frac{1}{a \cos t}$

$= -\frac{\operatorname{cosec}^2 t}{a \cos t}$ **(1)**

27. Differentiate the following with respect to x

$\sin^{-1}\left[\frac{2^{x+1} \cdot 3^x}{1+(36)^x}\right]$. **HOTS; All India 2013**

Firstly, put 6^x equal to $\tan\theta$, so that it becomes to some standard trigonometric function. Then, simplify the expression and then differentiate by chain rule.

Let $\quad y = \sin^{-1}\left[\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]$

$$= \sin^{-1}\left(\frac{2\cdot 2^x\cdot 3^x}{1+(6^2)^x}\right)$$

$$= \sin^{-1}\left[\frac{2\cdot 6^x}{1+(6^x)^2}\right] \quad \textbf{(1)}$$

Put $\tan\theta = 6^x \Rightarrow \theta = \tan^{-1}(6^x)$

Then, $\quad y = \sin^{-1}\left(\frac{2\cdot\tan\theta}{1+\tan^2\theta}\right)$

$$= \sin^{-1}(\sin 2\theta)\left[\because \sin\ 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right] \quad \textbf{(1)}$$

$$= 2\theta$$

$\Rightarrow y = 2\tan^{-1}(6^x)$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2}{1+(6^x)^2}\frac{d}{dx}(6^x)\left[\because \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}\right]$$

(1)

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{1+(6^x)^2}\cdot 6^x\cdot\log 6$$

$$= \left[\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]\log 6 \quad \textbf{(1)}$$

28. If $x=a\cos^3\theta$ and $y=a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta=\frac{\pi}{6}$. **All India 2013**

Given, $x = a\cos^3\theta$ and $y = a\sin^3\theta$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = 3a\cos^2\theta \frac{d}{d\theta}(\cos\theta)$$

$$= 3a\cos^2\theta \cdot (-\sin\theta)$$

$$= -3a\cos^2\theta \cdot \sin\theta \qquad \textbf{(1)}$$

and $$\frac{dy}{d\theta} = 3a\sin^2\theta \frac{d}{d\theta}(\sin\theta)$$

$$= 3a\sin^2\theta \cdot (\cos\theta) = 3a\sin^2\theta \cdot \cos\theta$$

Now, $$\frac{dy}{dx} = \left(\frac{dy/d\theta}{dx/d\theta}\right)$$

$$= \frac{3a\sin^2\theta \cdot \cos\theta}{-3a\cos^2\theta \cdot \sin\theta} = -\tan\theta \qquad \textbf{(1)}$$

Again, on differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\frac{d}{d\theta}(\tan\theta)\frac{d\theta}{dx}$$

$$= -\sec^2\theta \cdot \frac{d\theta}{dx}$$

$$= -\sec^2\theta \cdot \left(\frac{-1}{3a\cos^2\theta \cdot \sin\theta}\right)$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\theta \cdot \sin\theta} \qquad \textbf{(1)}$$

At

$$\theta = \frac{\pi}{6}, \left(\frac{d^2y}{dx^2}\right)_{\left(\theta = \frac{\pi}{6}\right)} = \frac{1}{3a\left(\cos\frac{\pi}{6}\right)^4\left(\sin\frac{\pi}{6}\right)}$$

$$= \frac{1}{3a\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right)}$$

$$= \frac{1}{3a\left(\frac{9}{16}\right)\left(\frac{1}{2}\right)} = \frac{32}{27a} \quad \textbf{(1)}$$

29. If $x \sin(a+y) + \sin a \cos(a+y) = 0$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$. **All India 2013**

To prove, $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

Given, $x \sin(a+y) + \sin a \cos(a+y) = 0$

$$\Rightarrow \quad x = \frac{-\sin a \cos(a+y)}{\sin(a+y)} \quad \textbf{(1)}$$

On differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{-\left[\sin(a+y)\frac{d}{dy}\{\sin a \cos(a+y)\} - \sin a \cos(a+y)\frac{d}{dy}\{\sin(a+y)\}\right]}{\sin^2(a+y)}$$

[using quotient rule]

$$= \left\{\frac{\sin(a+y)\cdot \sin a \sin(a+y) + \sin a \cos(a+y)\cos(a+y)}{\sin^2(a+y)}\right\} \textbf{(1)}$$

$$= \frac{\sin a}{\sin^2(a+y)}\{\sin^2(a+y) + \cos^2(a+y)\}$$

$$= \frac{\sin a}{\sin^2(a+y)}\cdot 1 \quad [\because \sin^2\theta + \cos^2\theta = 1] \, \textbf{(1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$ **Hence proved. (1)**

30. If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$ or $\frac{dy}{dx} = \frac{\log x}{\{\log(xe)\}^2}$.

All India 2013, 2011, 2010

> Firstly, take log on both sides and convert it into $y = f(x)$ form. Then, differentiate both sides by quotient rule to get required result.

To prove, $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Given $x^y = e^{x-y}$

On taking log both sides, we get

$$y \log_e x = (x-y)\log_e e \quad \textbf{(1)}$$

$\Rightarrow$ $y \log_e x = x - y$ $\quad [\because \log_e e = 1]$

$\Rightarrow$ $y(1+\log x) = x$

$\Rightarrow$ $$y = \frac{x}{1+\log x} \quad \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{v}{u}\right) = \frac{u\frac{dv}{dx} - v\frac{du}{dx}}{u^2}\right] \textbf{(1)}$$

$$= \frac{1+\log x - x \cdot \frac{1}{x}}{(1+\log x)^2}$$

$$= \frac{1 + \log x - 1}{(1 + \log x)^2}$$

$$= \frac{\log x}{(1 + \log x)^2}$$ **Hence proved. (1)**

Also, it can be written as

$$\frac{dy}{dx} = \frac{\log x}{(\log_e e + \log x)^2} \quad [-1\log_e e = 1]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log x}{\{\log(ex)\}^2}$$

31. If $y^x = e^{y-x}$, then prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.

All India 2013

To prove, $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$

Given that $y^x = e^{y-x}$

On taking log both sides, we get

$$\log y^x = \log e^{(y-x)}$$

$$\Rightarrow \quad x \log y = (y-x) \log e$$

$$\Rightarrow \quad x \log y = y - x \qquad [\because \log e = 1] \quad \text{...(i)} \; \mathbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$x \cdot \frac{d}{dx}(\log y) + \log y \frac{d}{dx}(x) = \frac{d}{dx}(y) - \frac{d}{dx}(x)$$

$$\Rightarrow \quad x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \log y \cdot 1 = \frac{dy}{dx} - 1$$

$$\Rightarrow \quad (1 + \log y) = \frac{dy}{dx}\left(1 - \frac{x}{y}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(1+\log y)}{(y-x)} \qquad \text{...(ii)} \; \mathbf{(1)}$$

On putting the value of x from Eq. (i) in Eq. (ii), we get

$$\frac{dy}{dx} = \frac{y(1+\log y)}{y - \left(\frac{y}{1+\log y}\right)} = \frac{y(1+\log y)^2}{(y + y\log y - y)} \qquad \mathbf{(1)}$$

$$= \frac{y(1+\log y)^2}{y \log y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y} \qquad \mathbf{(1)}$$

Hence proved.

32. If $(\cos x)^y = (\cos y)^x$, then find $\frac{dy}{dx}$.

HOTS; Delhi 2012

Firstly, take log on both sides, then differentiate both sides by product rule.

Given, $(\cos x)^y = (\cos y)^x$

On taking log both sides, we get

$$\log (\cos x)^y = \log (\cos y)^x$$

$$\Rightarrow \quad y \log (\cos x) = x \log(\cos y)$$

$$[\because \log x^n = n \log x] \quad \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$y \cdot \frac{d}{dx} \log (\cos x) + \log \cos x \cdot \frac{d}{dx}(y)$$

$$= x \frac{d}{dx} \log (\cos y) + \log (\cos y) \frac{d}{dx}(x)$$

$$\left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right] \textbf{(1)}$$

$$\Rightarrow \quad y \cdot \frac{1}{\cos x} \frac{d}{dx}(\cos x) + \log (\cos x) \frac{dy}{dx}$$

$$= x \cdot \frac{1}{\cos y} \frac{d}{dx}(\cos y) + \log \cos y \cdot 1$$

$$\Rightarrow \quad y. \frac{1}{\cos x}(-\sin x) + \log(\cos x) \cdot \frac{dy}{dx}$$

$$= x \frac{1}{\cos y}(-\sin y) \cdot \frac{dy}{dx} + \log(\cos y) \cdot 1 \; \textbf{(1)}$$

$$\Rightarrow -y \tan x + \log (\cos x) \frac{dy}{dx} = -x \tan y \frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow [x \tan y + \log (\cos x)] \frac{dy}{dx} = \log(\cos y) + y \tan x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log(\cos y) + y \tan x}{x \tan y + \log(\cos x)} \qquad \textbf{(1)}$$

33. If $\sin y = x\sin(a + y)$, then prove that

$$\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}.$$

HOTS; Delhi 2012

> In the given expression, we collect all the terms of y on RHS and a term x on LHS and then differentiate with respect to y on both sides to get required result.

Given, $\sin y = x \sin(a + y)$

$$\Rightarrow \quad x = \frac{\sin y}{\sin(a + y)}$$

On differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\sin(a + y).\frac{d}{dy}(\sin y) - \sin y.\frac{d}{dy}\sin(a + y)}{\sin^2(a + y)}$$

[by using quotient rule] **(1½)**

$$= \frac{\sin(a + y)\cos y - \sin y\cos(a + y)}{\sin^2(a + y)}$$

$$= \frac{\sin(a + y - y)}{\sin^2(a + y)}$$ **(1½)**

[$\because \sin A \cos B - \cos A \sin B = \sin(A - B)$]

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin a}{\sin^2(a + y)}$$

Hence, $$\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$$ **(1)**

Hence proved.

NOTE As the result is in y form, so we consider here x as a dependent variable.

34. If $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$, then show that $\frac{dy}{dx} = \frac{-y}{x}$.

All India 2012

Given, $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$

To show, $\frac{dy}{dx} = \frac{-y}{x}$

Now, $x = (a^{\sin^{-1} t})^{1/2}$

On differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = \frac{1}{2}(a^{\sin^{-1} t})^{-1/2} \frac{d}{dt}(a^{\sin^{-1} t})$$

$$\left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

$$= \frac{1}{2}(a^{\sin^{-1} t})^{-1/2}\, a^{\sin^{-1} t} \log a \frac{d}{dt}(\sin^{-1} t)$$

$$\left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$$

$$= \frac{1}{2}(a^{\sin^{-1} t})^{-1/2}\, a^{\sin^{-1} t} \log a \cdot \frac{1}{\sqrt{1-t^2}}$$

$$= \frac{1}{2}(a^{\sin^{-1} t})^{1/2} \log a \cdot \frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{\frac{1}{2}\sqrt{a^{\sin^{-1} t}} \cdot \log a}{\sqrt{1-t^2}} \quad \text{...(i)}$$ **(1½)**

And $y = (a^{\cos^{-1} t})^{1/2}$

On differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = \frac{1}{2}(a^{\cos^{-1} t})^{-1/2} \frac{d}{dt}(a^{\cos^{-1} t})$$

$$\left[\because \frac{d}{dx} x^n = nx^{n-1}\right]$$

$$= \frac{1}{2}(a^{\cos^{-1} t})^{-1/2} a^{\cos^{-1} t} \log a \frac{d}{dt}(\cos^{-1} t)$$

$$= \frac{1}{2}(a^{\cos^{-1} t})^{1/2} \log a \cdot \frac{(-1)}{\sqrt{1-t^2}}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-\frac{1}{2}\sqrt{a^{\cos^{-1} t}} \cdot \log a}{\sqrt{1-t^2}} \qquad \ldots\text{(ii)} \quad \mathbf{(1\frac{1}{2})}$$

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-\frac{1}{2}\sqrt{a^{\cos^{-1} t}} \log a}{\sqrt{1-t^2}}\right)}{\left(\frac{\frac{1}{2}\sqrt{a^{\sin^{-1} t}} \log a}{\sqrt{1-t^2}}\right)}$$

$$= -\frac{\sqrt{a^{\cos^{-1} t}}}{\sqrt{a^{\sin^{-1} t}}} = -\frac{y}{x} \qquad \mathbf{(1)}$$

$\left[\because \text{ given } \sqrt{a^{\cos^{-1} t}} = y \text{ and } \sqrt{a^{\sin^{-1} t}} = x\right]$

Hence proved.

35. Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ w.r.t. x.

HOTS; All India 2012

Firstly, put $x = \tan\theta$ and convert y in terms of θ, then put $\theta = \tan^{-1} x$ and differentiate w.r.t. x.

Let $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$, we get

$$y = \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta} \quad \textbf{(1)}$$

$$y = \tan^{-1}\left[\frac{\sec\theta - 1}{\tan\theta}\right] \quad [\because 1+\tan^2\theta = \sec^2\theta]$$

$$= \tan^{-1}\left[\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right]$$

$$\left[\because \sec\theta = \frac{1}{\cos\theta} \text{ and } \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$\left[\because 1-\cos x = 2\sin^2\frac{x}{2} \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} \quad [\because \tan^{-1}(\tan\phi) = \phi]$$

$$\Rightarrow \quad y = \frac{\tan^{-1} x}{2} \quad [\because \theta = \tan^{-1} x] \ \textbf{(1½)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1+x^2} \left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \right]$$

Hence, $\frac{dy}{dx} = \frac{1}{2(1+x^2)}$ **(1½)**

36. If $y = (\tan^{-1} x)^2$, then show that

$$(x^2+1)^2 \frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$$

Delhi 2012

Given, $y = (\tan^{-1} x)^2$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1} x . \frac{1}{1+x^2} \quad \left[\because \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\tan^{-1} x}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = 2\tan^{-1} x$$ **(1½)**

Again, differentiating both sides w.r.t. x, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}(1+x^2) = \frac{d}{dx}(2\tan^{-1} x)$$ **(1)**

$$\Rightarrow \quad (1+x^2).\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 2x = \frac{2}{1+x^2}$$

$$\left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \quad (1+x^2)^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}(1+x^2) = 2$$

$$\Rightarrow \quad (1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2$$ **(1½)**

Hence proved.

37. If $y = x^{\sin x - \cos x} + \frac{x^2-1}{x^2+1}$, then find $\frac{dy}{dx}$.

Delhi 2012C

Given, $y = x^{\sin x - \cos x} + \frac{x^2 - 1}{x^2 + 1}$

Let $u = x^{\sin x - \cos x}$

and $v = \frac{x^2 - 1}{x^2 + 1}$

Consider $u = x^{\sin x - \cos x}$

On taking log both sides, we get

$$\log u = (\sin x - \cos x) \cdot \log x$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = (\sin x - \cos x) \cdot \frac{1}{x} + \log x \cdot (\cos x + \sin x)$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x - \cos x}$$

$$\left[\frac{\sin x - \cos x}{x} + \log x \cdot (\cos x + \sin x)\right]$$

(1½)

Now, consider $v = \frac{x^2 - 1}{x^2 + 1} = 1 - \frac{2}{x^2 + 1}$

On differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx} = 0 - \frac{(x^2 + 1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2}$$

$$\Rightarrow \frac{dv}{dx} = -\left[\frac{0 - 2 \cdot 2x}{(x^2 + 1)^2}\right] = \frac{4x}{(x^2 + 1)^2} \quad \textbf{(1½)}$$

Now, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$$= x^{\sin x - \cos x}\left[\frac{\sin x - \cos x}{x}\right.$$

$$\left. + \log x(\cos x + \sin x) + \frac{4x}{(x^2 + 1)^2}\right] \quad \textbf{(1)}$$

38. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, then find $\frac{d^2y}{dx^2}$ and $\frac{d^2y}{dt^2}$. **Delhi 2012C**

Given, $x = a(\cos t + t\sin t)$

On differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = a(-\sin t + 1 \cdot \sin t + t\cos t) \quad \textbf{(1)}$$

$$= at\cos t$$

Also given, $y = a(\sin t - t\cos t)$

On differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = a(\cos t - \cos t \cdot 1 + t\sin t) = at\sin t \quad ...(i) \quad \textbf{(1)}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at\sin t}{at\cos t} = \tan t$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(\tan t)\frac{dt}{dx} = \sec^2 t \frac{1}{dx/dt}$$

$$= \frac{\sec^2 t}{at\cos t} = \frac{\sec^3 t}{at} \quad \textbf{(1)}$$

Also,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(at\sin t)$$

$$= a(\sin t + t\cos t) \quad \textbf{(1)}$$

39. Find $\frac{dy}{dx}$, when $y = x^{\cot x} + \frac{2x^2 - 3}{x^2 + x + 2}$.

All India 2012C

Given, $y = x^{\cot x} + \frac{2x^2 - 3}{x^2 + x + 2}$

Let $u = x^{\cot x}$ and $v = \frac{2x^2 - 3}{x^2 + x + 2}$

$$x^2+x+2$$

Consider $u = x^{\cot x}$

On taking log both sides, we get

$$\log u = \cot x \log x$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = \cot x \cdot \frac{1}{x} - \operatorname{cosec}^2 x \cdot \log x$$

$$\Rightarrow \quad \frac{du}{dx} = u\left(\frac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x\right)$$

$$= x^{\cot x}\left(\frac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x\right) \quad \mathbf{(1½)}$$

Now, consider $v = \dfrac{2x^2-3}{x^2+x+2}$

On differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx} = \frac{(x^2+x+2)(4x)-(2x^2-3)(2x+1)}{(x^2+x+2)^2}$$

$$= \frac{4x^3+4x^2+8x-4x^3-2x^2+6x+3}{(x^2+x+2)^2}$$

$$= \frac{2x^2+14x+3}{(x^2+x+2)^2} \quad \mathbf{(1½)}$$

Now, $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$

$$= x^{\cot x}\left(\frac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x\right)$$

$$+ \frac{2x^2+14x+3}{(x^2+x+2)^2} \quad \mathbf{(1)}$$

40. If $x = \cos t + \log \tan \frac{t}{2}$ and $y = \sin t$, then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. **All India 2012C**

Given, $x = \cos t + \log \tan\left(\frac{t}{2}\right)$

On differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = -\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2}$$

$$= -\sin t + \frac{\cos(t/2)}{\sin(t/2)} \cdot \frac{1}{\cos^2\left(\frac{t}{2}\right)} \cdot \frac{1}{2}$$

$$= -\sin t + \frac{1}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}$$

$$= -\sin t + \frac{1}{\sin t} = \frac{-\sin^2 t + 1}{\sin t} = \frac{\cos^2 t}{\sin t} \quad \textbf{(1)}$$

$$\left[\because 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta\right]$$

Also given, $y = \sin t$

On differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = \cos t$$

Again, differentiating both sides w.r.t. t, we get

$$\frac{d^2y}{dt^2} = -\sin t \quad \textbf{(1)}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{\frac{\cos^2 t}{\sin t}} = \frac{\sin t}{\cos t} = \tan t$$

On differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{\sin t}{\cos^2 t} = \sec^4 t \sin t$$

At $t = \frac{\pi}{4}$, $\frac{d^2y}{dt^2} = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$ **(1)**

At $t = \frac{\pi}{4}$, $\frac{d^2y}{dx^2} = \sec^4\frac{\pi}{4} \cdot \sin\frac{\pi}{4}$

$$= (\sqrt{2})^4 \cdot \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \quad \textbf{(1)}$$

41. If $x\sqrt{1+y} + y\sqrt{1+x} = 0, (x \neq y)$, then prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

HOTS; Foreign 2012; Delhi 2011C

Firstly, solve the given equation and convert it into $y = f(x)$ form. Then, differentiate to get the required result.

To prove, $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

Given equation is $x\sqrt{1+y} + y\sqrt{1+x} = 0$,

where $x \neq y$, we first convert the given equation into $y = f(x)$ form.

So, $$x\sqrt{1+y} = -y\sqrt{1+x}$$

On squaring both sides, we get

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow \quad x^2 + x^2y = y^2 + y^2x$$

$$\Rightarrow \quad x^2 - y^2 = y^2x - x^2y$$

$\Rightarrow \quad (x-y)(x+y) = -xy(x-y)$

$[\because a^2 - b^2 = (a-b)(a+b)]$

$\Rightarrow \quad (x-y)(x+y) + xy(x-y) = 0$

$\Rightarrow \quad (x-y)(x+y+xy) = 0$

$\therefore$ Either $x - y = 0$ or $x + y + xy = 0$

Now, $\quad x - y = 0 \Rightarrow x = y$

But it is given that $x \neq y$.

So, we get a contradiction.

$\Rightarrow \quad x - y = 0$ is rejected. **(1)**

$\therefore \quad y + xy + x = 0 \Rightarrow y(1+x) = -x$

$\Rightarrow \quad y = \frac{-x}{1+x}$ ...(i) **(1)**

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+x) \times \frac{d}{dx}(-x) - (-x) \times \frac{d}{dx}(1+x)}{(1+x)^2} \quad \textbf{(1)}$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1+x)(-1) + x(1)}{(1+x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1-x+x}{(1+x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{(1+x)^2} \quad \textbf{(1)}$$

Hence proved

42. If $x = \tan\left(\frac{1}{a}\log y\right)$, then show that

$(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0.$ **All India 2011**

Given, $x = \tan\left(\frac{1}{a}\log y\right)$

$$\Rightarrow \quad \tan^{-1} x = \frac{1}{a}\log y$$

$$[\because \tan\theta = a \Rightarrow \theta = \tan^{-1} a]$$

$$\Rightarrow \quad a\tan^{-1} x = \log y$$

On differentiating both sides w.r.t. x, we get

$$a \times \frac{1}{1+x^2} = \frac{1}{y}\cdot\frac{dy}{dx} \qquad \textbf{(1)}$$

$$\left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = ay \qquad \textbf{(1)}$$

Again, on differentiating both sides w.r.t. x, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(1+x^2) = \frac{d}{dx}(ay)$$

$$\left[\because \frac{d}{dx}(u\cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}\right] \textbf{(1)}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot(2x) = a\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - a\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0 \qquad \textbf{(1)}$$

43. Differentiate $x^{x\cos x} + \frac{x^2+1}{x^2-1}$ w.r.t. x.

Delhi 2011

Let $u = x^{x \cos x}$ and $v = \frac{x^2+1}{x^2-1}$, then $y = u + v$.

Now, find $\frac{du}{dx}$ and $\frac{dv}{dx}$. Then, put these values in $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Let $\quad y = x^{x \cos x} + \frac{x^2+1}{x^2-1}$

Again let $\quad u = x^{x \cos x}$ and $v = \frac{x^2+1}{x^2-1}$

Then, $\quad y = u + v$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)} \ \mathbf{(1)}$$

Now, $\quad u = x^{x \cos x}$

On taking log both sides, we get

$$\log u = \log x^{x \cos x}$$

$$\Rightarrow \log u = (x \cos x) \cdot \log x \quad [\because \log m^n = n \log m]$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = (x \cos x) \times \frac{d}{dx}(\log x) + \log x \times \frac{d}{dx}(x \cos x)$$

[by product rule]

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x \cos x \times \frac{1}{x} + \log x \cdot [-x \sin x + \cos x]$$

$$\left[\begin{aligned} \because \frac{d}{dx}(x \cos x) &= x \times \frac{d}{dx} \cos x + \cos x \frac{d}{dx} x \\ &= x(-\sin x) + \cos x \cdot 1 \\ &= -x \sin x + \cos x \end{aligned}\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x - x \log x \sin x + \log x \cos x$$

$$\Rightarrow \frac{du}{dx} = u\ [\cos x - x \log x \sin x + \log x \cos x]$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x}[\cos x - x \log x \sin x + \log x \cos x]\ [\because u = x^{x\cos x}]$$ **(1½)**

and $v = \dfrac{x^2+1}{x^2-1}$

On differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \frac{(x^2-1)\times\frac{d}{dx}(x^2+1)-(x^2+1)\times\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

[by quotient rule]

$$\Rightarrow \frac{dv}{dx} = \frac{(x^2-1)\cdot 2x-(x^2+1)\cdot 2x}{(x^2-1)^2}$$

$$= \frac{2x^3-2x-2x^3-2x}{(x^2-1)^2}$$

$$\Rightarrow \frac{dv}{dx} = \frac{-4x}{(x^2-1)^2}$$

On putting values of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ in Eq. (i), we get

$$\frac{dy}{dx} = x^{x\cos x}[\cos x - x\log x \sin x + \log x \cos x] - \frac{4x}{(x^2-1)^2}$$ **(1½)**

44. If $x = a(\theta - \sin\theta)$, $y = a(1+\cos\theta)$, then find $\dfrac{d^2y}{dx^2}$.

Delhi 2011

Here, we use chain rule, i.e. if $y = f_1(\theta)$ and $x = f_2(\theta)$, then $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$ to get required value.

Given, $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \quad \left[\because \frac{d}{d\theta}\sin\theta = \cos\theta\right]$$

and $$\frac{dy}{d\theta} = -a\sin\theta \left[\because \frac{d}{d\theta}\cos\theta = -\sin\theta\right] \textbf{(1)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{-a\sin\theta}{a(1 - \cos\theta)} = \frac{-\sin\theta}{1 - \cos\theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\cot\frac{\theta}{2} \qquad \textbf{(1)}$$

$$\left[\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \text{ and } 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\cot\frac{\theta}{2}\right)$$

$$= \frac{d}{d\theta}\left(-\cot\frac{\theta}{2}\right) \times \frac{d\theta}{dx}$$

$$\left[\because \frac{d}{dx}[f(\theta)] = \frac{d}{d\theta}f(\theta) \times \frac{d\theta}{dx}\right]$$

$$= \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2} \times \frac{1}{a(1-\cos\theta)} \qquad \textbf{(1)}$$

$$\left[\because \frac{d}{d\theta}\cot\theta = -\operatorname{cosec}^2\theta\right]$$

$$= \frac{1}{2a}\operatorname{cosec}^2\frac{\theta}{2} \times \frac{1}{2\sin^2\frac{\theta}{2}}$$

$$\left[\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2} \qquad \textbf{(1)}$$

45. Prove that

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right] = \sqrt{a^2-x^2}.$$

Foreign 2011

To prove $\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$

$$=\sqrt{a^2-x^2}$$

$$\text{LHS}=\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$=\left[\frac{x}{2}\times\frac{d}{dx}\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\right.$$

$$\left.\times\frac{d}{dx}\left(\frac{x}{2}\right)+\frac{a^2}{2}\times\frac{d}{dx}\sin^{-1}\frac{x}{a}\right] \textbf{(1)}$$

$$\left[\because \frac{d}{dx}(u\cdot v)=u\frac{dv}{dx}+v\frac{du}{dx}\right]$$

$$=\frac{x}{2}\cdot\frac{1}{2\sqrt{a^2-x^2}}\frac{d}{dx}(a^2-x^2)$$

$$+\sqrt{a^2-x^2}\cdot\frac{1}{2}+\frac{a^2}{2}\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$=\left[\frac{x}{2}\cdot\frac{-2x}{2\sqrt{a^2-x^2}}+\sqrt{a^2-x^2}\cdot\frac{1}{2}\right.$$

$$\left.+\frac{a^2}{2}\times\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}\right] \textbf{(1)}$$

$$=\frac{-x^2}{2\sqrt{a^2-x^2}}+\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2a}\cdot\frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}}$$

$$= \frac{-x^2}{2\sqrt{a^2 - x^2}} + \frac{\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2a} \times \frac{a}{\sqrt{a^2 - x^2}}$$

$$= \frac{-x^2}{2\sqrt{a^2 - x^2}} + \frac{\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2\sqrt{a^2 - x^2}} \quad \textbf{(1)}$$

$$= \frac{-x^2 + (a^2 - x^2) + a^2}{2\sqrt{a^2 - x^2}}$$

$$= \frac{2a^2 - 2x^2}{2\sqrt{a^2 - x^2}} = \frac{2(a^2 - x^2)}{2\sqrt{a^2 - x^2}}$$

$$= \frac{a^2 - x^2}{(a^2 - x^2)^{1/2}} = (a^2 - x^2)^{1/2} = \sqrt{a^2 - x^2}$$

$= \text{RHS}$ **Hence proved. (1)**

46. If $y = \log [x + \sqrt{x^2 + 1}]$, then prove that

$$(x^2 + 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

Foreign 2011

Do same as Que 24.
[**Hint** put $a = 1$ in Que. 24]

47. If $\log (\sqrt{1 + x^2} - x) = y\sqrt{1 + x^2}$, then show that $(1 + x^2)\frac{dy}{dx} + xy + 1 = 0.$

All India 2011C

To prove, $(1+x^2)\frac{dy}{dx}+xy+1=0$

Given, $\log(\sqrt{1+x^2}-x)=y\sqrt{1+x^2}$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1+x^2}-x}\frac{d}{dx}[\sqrt{1+x^2}-x]$$

$$=y\frac{d}{dx}\sqrt{1+x^2}+\sqrt{1+x^2}\frac{dy}{dx} \quad \textbf{(1)}$$

[by chain rule]

$$\Rightarrow \frac{1}{\sqrt{1+x^2}-x}\left[\frac{1}{2\sqrt{1+x^2}}\frac{d}{dx}(x^2)-1\right]$$

$$=\frac{y}{2\sqrt{1+x^2}}\frac{d}{dx}(1+x^2)+\sqrt{1+x^2}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{\sqrt{1+x^2}-x}\left[\frac{2x}{2\sqrt{1+x^2}}-1\right]$$

$$=y\times\frac{2x}{2\sqrt{1+x^2}}+\sqrt{1+x^2}\cdot\frac{dy}{dx} \quad \textbf{(1½)}$$

$$\Rightarrow \frac{1}{\sqrt{1+x^2}-x}\left[\frac{x-\sqrt{1+x^2}}{\sqrt{1+x^2}}\right]$$

$$=\frac{xy}{\sqrt{1+x^2}}+\sqrt{1+x^2}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{-1}{\sqrt{1+x^2}}=\frac{xy+(1+x^2)\frac{dy}{dx}}{\sqrt{1+x^2}}$$

$$\Rightarrow -1=xy+(1+x^2)\frac{dy}{dx}$$

$$\Rightarrow (1+x^2)\frac{dy}{dx}+xy+1=0 \quad \textbf{(1½)}$$

Hence proved.

48. If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, then find $\frac{d^2y}{dx^2}$.

All India 2011C

Given,

$x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dx}{d\theta} = a(1 + \cos\theta) \quad \left[\because \frac{d}{d\theta}\sin\theta = \cos\theta\right]$$

and $$\frac{dy}{d\theta} = a\sin\theta \quad \left[\because \frac{d}{d\theta}\cos\theta = -\sin\theta\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$\left[\begin{array}{l}\because \quad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \\ \text{and } 1 + \cos x = 2\cos^2\frac{x}{2}\end{array}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \tan\frac{\theta}{2}$$ **(1)**

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\tan\frac{\theta}{2}\right) \quad \textbf{(1/2)}$$

$$\left[\because \frac{d}{dx} f(\theta) = \frac{d}{d\theta} f(\theta) \cdot \frac{d\theta}{dx}\right]$$

$$= \frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right) \times \frac{d\theta}{dx} = \sec^2\frac{\theta}{2} \cdot \frac{d}{d\theta}\left(\frac{\theta}{2}\right) \cdot \frac{d\theta}{dx}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2} \times \frac{1}{a(1+\cos\theta)} \quad \textbf{(1)}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2} \times \frac{1}{a \times 2\cos^2\frac{\theta}{2}}$$

$$\left[1 + \cos x = 2\cos^2\frac{x}{2}\right]$$

$$= \frac{1}{4a}\sec^4\frac{\theta}{2} \quad \textbf{(1½)}$$

49. If $y = a\sin x + b\cos x$, then prove that

$$y^2 + \left(\frac{dy}{dx}\right)^2 = a^2 + b^2.$$

All India 2011C; HOTS

Firstly, we differentiate the given expression with respect to x and get first derivative of y. Now, put the value of y and first derivative of y in LHS of given expression and then solve it and get required RHS.

To prove, $y^2 + \left(\frac{dy}{dx}\right)^2 = a^2 + b^2$ ...(i)

Given, $y = a \sin x + b \cos x$ ...(ii)

On differentiating both sides of Eq. (ii) w.r.t. x, we get

$$\frac{dy}{dx} = a \cos x - b \sin x \quad \textbf{(1)}$$

Now, we take LHS of Eq. (i), we get

$$\text{LHS} = y^2 + \left(\frac{dy}{dx}\right)^2$$

On putting the value of y and dy/dx, we get

$$\text{LHS} = (a \sin x + b \cos x)^2 + (a \cos x - b \sin x)^2$$

$$= a^2 \sin^2 x + b^2 \cos^2 x + 2ab \sin x \cos x + a^2 \cos^2 x + b^2 \sin^2 x - 2ab \sin x \cos x \quad \textbf{(1½)}$$

$$[\because (a \pm b)^2 = a^2 + b^2 \pm 2ab]$$

$$= a^2 \sin^2 x + b^2 \cos^2 x + a^2 \cos^2 x + b^2 \sin^2 x$$

$$= a^2 (\sin^2 x + \cos^2 x) + b^2 (\sin^2 x + \cos^2 x)$$

$$= a^2 + b^2 \quad [\because \sin^2 x + \cos^2 x = 1]$$

= RHS **Hence proved. (1½)**

50. If $x = a(\cos\theta + \theta\sin\theta)$

and $y = a(\sin\theta - \theta\cos\theta)$, then find $\frac{d^2y}{dx^2}$.

All India 2011C, 2008

Do same as Que 38 $\left[\textbf{Ans. } 7\dfrac{\sec^3\theta}{a\theta}\right]$

51. If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, then find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{3}$.

Delhi 2011C

Given, $x = a(\theta - \sin\theta)$

and $y = a(1 + \cos\theta)$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \quad \text{and} \quad \frac{dy}{d\theta} = -a\sin\theta \quad \textbf{(1)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1-\cos\theta)} \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{a \times 2\sin^2\frac{\theta}{2}}$$

$$\left[\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \text{ and } 1 - \cos x = 2\sin^2\frac{x}{2}\right] \textbf{(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\cot\frac{\theta}{2}$$

On putting $\theta = \dfrac{\pi}{3}$, we get

$$\left[\frac{dy}{dx}\right]_{\theta = \frac{\pi}{3}} = -\cot\frac{\pi}{6} = -\sqrt{3}$$

$$\left[\because \cot\frac{\pi}{6} = \sqrt{3}\right] \textbf{(1)}$$

Hence, $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{3}$ is $-\sqrt{3}$.

52. If $y = (\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$,

then find $\frac{dy}{dx}$. ,

All India 2010C

Firstly, take log on both sides and then differentiate to get the value of $\frac{dy}{dx}$.

Given, $y = (\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

On taking log both sides, we get

$$\log y = \log (\sin x - \cos x)^{(\sin x - \cos x)}$$

$$\Rightarrow \log y = (\sin x - \cos x) \cdot \log (\sin x - \cos x)$$

$[\because \log m^n = n \log m]$ **(1)**

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = (\sin x - \cos x) \times \frac{d}{dx} \log (\sin x - \cos x)$$

$$+ \log (\sin x - \cos x) \times \frac{d}{dx} (\sin x - \cos x)$$

[by product rule]

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x)\cdot\frac{1}{(\sin x - \cos x)}$$

$$\cdot\frac{d}{dx}(\sin x - \cos x) + \log(\sin x - \cos x)$$

$$\cdot(\cos x + \sin x) \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x)\frac{1}{(\sin x - \cos x)}$$

$$(\cos x + \sin x) + \log(\sin x - \cos x)$$

$$\cdot(\cos x + \sin x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (\cos x + \sin x) + (\cos x + \sin x)$$

$$\cdot[\log(\sin x - \cos x)]$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = (\cos x + \sin x)$$

$$[1 + \log(\sin x - \cos x)] \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = y(\cos x + \sin x)$$

$$[1 + \log(\sin x - \cos x)]$$

$$\therefore \quad \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}$$

$$\cdot(\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

(1)

53. If $y = \cos^{-1}\left[\frac{2x - 3\sqrt{1 - x^2}}{\sqrt{13}}\right]$, then find $\frac{dy}{dx}$.

All India 2010C

In the given expression, put $x = \sin\theta$ and simplify the resulting expression, then differentiate it.

Given, $y = \cos^{-1}\left[\dfrac{2x - 3\sqrt{1-x^2}}{\sqrt{13}}\right]$

Put $\quad x = \sin\theta$, then $\theta = \sin^{-1} x$

$$\left[\begin{array}{l} \because \text{ for } \sqrt{a^2 - x^2}, \text{ we put } x = a\sin\theta \\ \therefore \text{ for } \sqrt{1-x^2}, \text{ we put } x = \sin\theta \end{array}\right] \quad \textbf{(1)}$$

$\therefore \quad y = \cos^{-1}\left[\dfrac{2\sin\theta - 3\sqrt{1-\sin^2\theta}}{\sqrt{13}}\right]$

$\Rightarrow \quad y = \cos^{-1}\left[\dfrac{2\sin\theta - 3\cos\theta}{\sqrt{13}}\right]$

$[\because \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta]$

$\Rightarrow y = \cos^{-1}\left[\dfrac{2}{\sqrt{13}}\sin\theta - \dfrac{3}{\sqrt{13}}\cos\theta\right]$

Now, let $\frac{2}{\sqrt{13}} = \cos\alpha$ and $\frac{3}{\sqrt{13}} = \sin\alpha$ **(1½)**

$$\left[\because \sin^2\alpha + \cos^2\alpha = \left(\frac{3}{\sqrt{13}}\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2 = \frac{9}{13} + \frac{4}{13} = \frac{13}{13} = 1\right]$$

$\therefore\quad y = \cos^{-1}[\sin\theta\cos\alpha - \cos\theta\sin\alpha]$

$\Rightarrow\quad y = \cos^{-1}\sin(\theta - \alpha)$

$[\because \sin\theta\cos\alpha - \cos\theta\sin\alpha = \sin(\theta - \alpha)]$

$$\Rightarrow\quad y = \cos^{-1}\cos\left[\frac{\pi}{2} - (\theta - \alpha)\right]$$

$$\left[\begin{array}{l}\because \sin x = \cos\left(\frac{\pi}{2} - x\right) \\ \text{here, } x = \theta - \alpha\end{array}\right]$$

$$\Rightarrow\quad y = \frac{\pi}{2} - \theta + \alpha \quad [\because \cos^{-1}(\cos\theta) = \theta]$$

$$\Rightarrow\quad y = \frac{\pi}{2} - \sin^{-1}x + \alpha \quad [\because \theta = \sin^{-1}x]$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 0 - \frac{1}{\sqrt{1-x^2}} + 0$$

$$\left[\because \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\quad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$ **(1½)**

54. If $y = (\cot^{-1}x)^2$, then show that

$$(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$$ **Delhi 2010C**

To show, $(x^2+1)^2 \frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$

Given, $y = (\cot^{-1} x)^2$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2 \cdot \cot^{-1} x \cdot \frac{d}{dx}(\cot^{-1} x)$$

$$\Rightarrow \quad \frac{dy}{dx} = 2\cot^{-1} x \times \frac{-1}{1+x^2}$$

$$\left[\because \frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}\right]$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = -2\cot^{-1} x \qquad \textbf{(1½)}$$

Again, differentiating both sides w.r.t. x, we get

$$(1+x^2) \times \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \times \frac{d}{dx}(1+x^2)$$

$$= \frac{d}{dx}(-2\cot^{-1} x)$$

$$\left[\because \frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$\Rightarrow (1+x^2) \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 2x = \frac{-2 \times (-1)}{1+x^2} \qquad \textbf{(1½)}$$

$$\left[\because \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}\right]$$

On multiplying both sides by $(1+x^2)$, we get

$$(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2 \qquad \textbf{(1)}$$

Hence proved.

55. If $y = \operatorname{cosec}^{-1} x, x > 1$, then show that

$$x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0.$$ **All India 2010**

To show, $x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0$

Given, $y = \operatorname{cosec}^{-1} x;\ x > 1$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}} \Rightarrow x\sqrt{x^2-1}\cdot\frac{dy}{dx} = -1 \quad \textbf{(1)}$$

Again, differentiating both sides w.r.t. x, we get

$$(x\sqrt{x^2-1})\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}(x\sqrt{x^2-1}) = \frac{d}{dx}(-1)$$

$$\left[\because \frac{d}{dx}(u\cdot v) = u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx}\right] \quad \textbf{(1)}$$

$$\Rightarrow x\sqrt{x^2-1}\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\left\{x\times\frac{d}{dx}\sqrt{x^2-1} + \sqrt{x^2-1}\times\frac{d}{dx}(x)\right\} = 0$$

$$\Rightarrow x\sqrt{x^2-1}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left\{\frac{x}{2\sqrt{x^2-1}}\frac{d}{dx}(x^2-1) + \sqrt{x^2-1}\times 1\right\} = 0$$

$$\Rightarrow x\sqrt{x^2-1}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left\{\frac{x\cdot 2x}{2\sqrt{x^2-1}} + \sqrt{x^2-1}\right\} = 0$$

$$\Rightarrow x\sqrt{x^2-1}\frac{d^2y}{dx^2}+\frac{dy}{dx}\left\{\frac{x^2}{\sqrt{x^2-1}}+\sqrt{x^2-1}\right\}=0$$

$$\Rightarrow x\sqrt{x^2-1}\frac{d^2y}{dx^2}+\frac{x^2}{\sqrt{x^2-1}}\frac{dy}{dx}+\sqrt{x^2-1}\frac{dy}{dx}=0$$

(1)

On multiplying both sides by $\sqrt{x^2-1}$, we get

$$x(x^2-1)\frac{d^2y}{dx^2}+x^2\frac{dy}{dx}+(x^2-1)\frac{dy}{dx}=0$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2}+(x^2+x^2-1)\frac{dy}{dx}=0$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2}+(2x^2-1)\frac{dy}{dx}=0$$ **(1)**

Hence proved.

56. If $y=\cos^{-1}\left(\frac{3x+4\sqrt{1-x^2}}{5}\right)$, then find $\frac{dy}{dx}$.

All India 2010

Given, $y = \cos^{-1}\left[\frac{3x + 4\sqrt{1-x^2}}{5}\right]$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$, we get **(1)**

$$\therefore \quad y = \cos^{-1}\left[\frac{3\sin\theta + 4\sqrt{1-\sin^2\theta}}{5}\right]$$

$$\Rightarrow \quad y = \cos^{-1}\left[\frac{3\sin\theta + 4\cos\theta}{5}\right]$$

$$[\because \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta]$$

$$\Rightarrow \quad y = \cos^{-1}\left[\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta\right]$$ **(1)**

Now, let $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$

$$\left[\because \sin^2\alpha + \cos^2\alpha = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1\right]$$

Then, we get

$$y = \cos^{-1}[\sin\theta\sin\alpha + \cos\theta\cos\alpha]$$

$$\Rightarrow \quad y = \cos^{-1}\cos(\theta - \alpha)$$

$$[\because \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos(\theta-\alpha)]$$

$$\Rightarrow \quad y = \theta - \alpha$$

$$\Rightarrow \quad y = \sin^{-1}x - \alpha \quad [\because \theta = \sin^{-1}x]$$ **(1)**

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - 0 \quad \left[\because \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right]$$

Hence, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ **(1)**

57. Show that the function defined as follows, is continuous at $x = 1$, $x = 2$ but not differentiable at $x = 2$

$$f(x) = \begin{cases} 3x - 2, & 0 < x \le 1 \\ 2x^2 - x, & 1 < x \le 2. \\ 5x - 4, & x > 2 \end{cases}$$

Delhi 2010

> Firstly, we will check the continuity of the given function at $x = 1, 2$ and then to check the differentiability of the function $f(x)$ at these points, find LHD and RHD. If LHD $\neq$ RHD, then function is not differentiable.

The given function is $f(x) = \begin{cases} 3x - 2, & 0 < x \le 1 \\ 2x^2 - x, & 1 < x \le 2 \\ 5x - 4, & x > 2 \end{cases}$

First, we show the continuity of above function at $x = 1$ and at $x = 2$.

Continuity at $x = 1$

$$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x - 2)$$

[put $x = 1 - h$, when $x \to 1$, $h \to 0$]

$$\Rightarrow \quad \text{LHL} = \lim_{h \to 0} [3(1 - h) - 2]$$

$$= \lim_{h \to 0} (3 - 3h - 2) = 1$$

$$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x^2 - x)$$

[put $x = 1 + h$, when $x \to 1$, $h \to 0$]

$$\Rightarrow \text{RHL} = \lim_{h \to 0} [2(1 + h)^2 - (1 + h)]$$

$$= \lim_{h \to 0} [2(1 + h^2 + 2h) - (1 + h)]$$

$$= \lim_{h \to 0} [2 + 2h^2 + 4h - 1 - h]$$

$$= \lim_{h \to 0} (2h^2 + 3h + 1) \Rightarrow \text{RHL} = 1$$

Also, from the given function, at $x = 1$

$$f(1) = 3(1) - 2 = 3 - 2 = 1$$

Since, $\text{LHL} = \text{RHL} = f(1)$

Hence, $f(x)$ is continuous at $x = 1$. **(1)**

Continuity at $x = 2$

$$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x^2 - x)$$

[put $x = 2 - h$, when $x \to 2, h \to 0$]

$$\Rightarrow \text{LHL} = \lim_{h \to 0} [2(2-h)^2 - (2-h)]$$

$$= \lim_{h \to 0} [2(4 + h^2 - 4h) - (2 - h)]$$

$$= \lim_{h \to 0} (8 + 2h^2 - 8h - 2 + h)$$

$\Rightarrow \text{LHL} = 8 - 2 = 6$ [put $h = 0$]

and $\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (5x - 4)$

[put $x = 2 + h$, when $x \to 2, h \to 0$]

$$\Rightarrow \quad \text{RHL} = \lim_{h \to 0} [5(2+h) - 4]$$

$$= \lim_{h \to 0} (10 + 5h - 4) = \lim_{h \to 0} (5h + 6)$$

$\Rightarrow \quad \text{RHL} = 6$

Also, from the given function, at $x = 2$.

$$f(2) = 2(2)^2 - 2$$

[for $f(2)$, put $x = 2$ in $f(x) = 2x^2 - x$]

$$= 8 - 2 = 6$$

Since, $\text{LHL} = \text{RHL} = f(2)$

So, $f(x)$ is continuous at $x = 2$. **(1)**

Hence, $f(x)$ is continuous at all indicated points.

Now, we verify differentiability of the given function at $x = 1$ and $x = 2$.

Differentiability at $x = 1$

$$\text{LHD} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{[3(1-h)-2]-[3-2]}{-h}$$

$$= \lim_{h \to 0} \frac{(1-3h)-(1)}{-h} = \lim_{h \to 0} \frac{-3h}{-h}$$

$\Rightarrow \quad \text{LHD} = 3$

$$\text{RHD} = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h \to 0} \frac{[2(1+h)^2-(1+h)]-[2-1]}{h}$$

$$= \lim_{h \to 0} \frac{[2+2h^2+4h-1-h]-1}{h}$$

$[\because (a+b)^2 = a^2+b^2+2ab]$

$$= \lim_{h \to 0} \frac{2h^2+3h}{h} = \lim_{h \to 0} (2h+3)$$

$\Rightarrow \quad \text{RHD} = 3$ [put $h = 0$]

Since, LHD = RHD

So, $f(x)$ is differentiable at $x = 1$. **(1)**

Differentiability at $x = 2$

$$\text{LHD} = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$\Rightarrow \text{LHD} = \lim_{h \to 0} \frac{[2(2-h)^2 - (2-h)] - [8-2]}{-h}$$

$$= \lim_{h \to 0} \frac{2(4 + h^2 - 4h) - (2-h) - 6}{-h}$$

$$= \lim_{h \to 0} \frac{h(2h-7)}{-h}$$

$$\Rightarrow \text{LHD} = 7$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{[5(2+h) - 4] - [8-2]}{h}$$

$$= \lim_{h \to 0} \frac{(6+5h) - (6)}{h}$$

$$= \lim_{h \to 0} \frac{5h}{h}$$

$$\Rightarrow \text{RHD} = 5$$

Since, LHD $\neq$ RHD

So, $f(x)$ is not differentiable at $x = 2$. **(1)**

Hence, $f(x)$ is continuous at $x = 1$ and $x = 2$ but not differentiable at $x = 2$.

58. If $y = e^{a \cos^{-1} x}, -1 \le x \le 1$, then show that

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0.$$

All India 2010

To show, $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Given, $y = e^{a\cos^{-1}x}, -1 \le x \le 1$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{a\cos^{-1}x} \cdot \frac{d}{dx}(a\cos^{-1}x)$$

[by chain rule]

$$\Rightarrow \quad \frac{dy}{dx} = e^{a\cos^{-1}x} \times \frac{-a}{\sqrt{1-x^2}}$$

$$\left[\because \frac{d}{dx}e^x = e^x, \frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = -ae^{a\cos^{-1}x}$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = -ay \quad \ldots(i)$$

$[\because e^{a\cos^{-1}x} = y$, given] **(1)**

Again, differentiating both sides of Eq. (i) w.r.t. x, we get

$$\sqrt{1-x^2} \times \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(\sqrt{1-x^2})$$

$$= \frac{d}{dx}(-ay) \quad [\because \text{by product rule}]$$

$$\Rightarrow \sqrt{1-x^2} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}(1-x^2)$$

$$= -a \cdot \frac{dy}{dx}$$

$$\Rightarrow \sqrt{1-x^2} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{-2x}{2\sqrt{1-x^2}} = -a \cdot \frac{dy}{dx}$$

$$\Rightarrow \sqrt{1-x^2} \frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}} \cdot \frac{dy}{dx} = -a \cdot \frac{dy}{dx} \quad \textbf{(1½)}$$

On multiplying both sides by $\sqrt{1-x^2}$, we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -a\sqrt{1-x^2} \cdot \frac{dy}{dx} \quad \text{...(ii)}$$

But from Eq. (i), we have

$$\sqrt{1-x^2}\frac{dy}{dx} = -ay \Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}} \quad \text{...(iii)} \ \textbf{(1/2)}$$

On putting the value of $\frac{dy}{dx}$ from Eq. (iii) in Eq. (ii), we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -a\sqrt{1-x^2} \times \frac{-ay}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$$

$$\therefore (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0 \quad \textbf{(1)}$$

Hence proved.

59. Find $\frac{dy}{dx}$, if $y = (\cos x)^x + (\sin x)^{1/x}$. **Delhi 2010**

Let $u = (\cos x)^x$ and $v = (\sin x)^{1/x}$. Now, take log on both sides of u and v and then differentiate with respect to x to get $\frac{du}{dx}$ and $\frac{dv}{dx}$. Further, put these values in equation $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Given, $y = (\cos x)^x + (\sin x)^{1/x}$

Let $u = (\cos x)^x$ and $v = (\sin x)^{1/x}$

Then, given equation becomes,

$$y = u + v$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots(i)$$

Now, $u = (\cos x)^x$

On taking log both sides, we get

$$\log u = \log (\cos x)^x$$

$$\Rightarrow \quad \log u = x \log (\cos x)$$

$$[\because \log m^n = n \log m]$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{d}{dx} \log (\cos x) + \log (\cos x) \cdot \frac{d}{dx}(x)$$

$$\left[\because \frac{d}{dx}(\log u) = \frac{1}{u}\frac{du}{dx}\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{\cos x}(-\sin x) + \log \cos x \cdot 1$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = -x \tan x + \log (\cos x)$$

$$\Rightarrow \quad \frac{du}{dx} = u\,[-x \tan x + \log \cos x]$$

$$\Rightarrow \quad \frac{du}{dx} = (\cos x)^x$$

$$[-x \tan x + \log \cos x] \qquad \text{...(ii)} \ \mathbf{(1)}$$

Also, $v = (\sin x)^{1/x}$

On taking log both sides, we get

$$\log v = \log (\sin x)^{1/x}$$

$$\Rightarrow \quad \log v = \frac{1}{x} \log \sin x$$

$$[\because \log m^n = n \log m]$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{1}{x} \cdot \frac{d}{dx} (\log \sin x) + \log \sin x \cdot \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = \frac{1}{x} \cdot \frac{1}{\sin x} \cdot \cos x + \log \sin x \left(-\frac{1}{x^2}\right)$$

$$\left[\begin{array}{l} \because \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2} \\ \text{and } \dfrac{d}{dx}(\log v) = \dfrac{1}{v} \dfrac{dv}{dx} \end{array}\right]$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = \frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}$$

$$\Rightarrow \frac{dv}{dx} = v\left(\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right)$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{1/x}\left[\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right] \quad \ldots\text{(iii)}$$

(1½)

On putting the value of $\frac{du}{dx}$ from Eq. (ii) and that of $\frac{dv}{dx}$ from Eq. (iii) in Eq. (i), we get

$$\frac{dy}{dx} = (\cos x)^x [-x \tan x + \log \cos x] + (\sin x)^{1/x}\left[\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right]$$ **(1½)**

60. If $y = e^x \sin x$, then prove that

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$

All India 2010C

Firstly, we find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and then put their values along with value of y in LHS of proven expression.

Given, $y = e^x \sin x$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(e^x)$$

[by using product rule]

$$\Rightarrow \quad \frac{dy}{dx} = e^x \cdot \cos x + \sin x \cdot e^x$$ **(1)**

$$\Rightarrow \quad \frac{dy}{dx} = e^x (\cos x + \sin x) \quad ...(ii)$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = e^x \cdot \frac{d}{dx}(\cos x + \sin x) + (\cos x + \sin x) \cdot \frac{d}{dx}(e^x)$$

[by using product rule]

$$\Rightarrow \quad \frac{d^2y}{dx^2} = e^x (-\sin x + \cos x) + (\cos x + \sin x) \cdot e^x$$

$$= e^x [-\sin x + \cos x + \cos x + \sin x]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2 \cos x \, e^x \quad ...(iii)$$ **(1)**

Now, we have to show that

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

On putting the values of $\frac{d^2y}{dx^2}$ from Eq. (iii), $\frac{dy}{dx}$ from Eq. (ii) and that of y from Eq. (i) on LHS, we get

$$\text{LHS} = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y \quad \textbf{(1)}$$

$$= 2e^x \cos x - 2e^x (\cos x + \sin x) + 2e^x \sin x$$

$$= 2e^x \cos x - 2e^x \cos x - 2e^x \sin x + 2e^x \sin x$$

$$= 0 = \text{RHS} \quad \textbf{(1)}$$

Hence proved.

61. If $y = (x)^x + (\sin x)^x$, then find $\frac{dy}{dx}$.

All India 2010C

Given, $y = (x)^x + (\sin x)^x$

Let $u = (x)^x$ and $v = (\sin x)^x$

$\therefore$ Given equation becomes, $y = u + v$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots\text{(i)}$$

Now, $u = x^x$ **(1)**

On taking log both sides, we get

$$\log u = \log x^x \Rightarrow \log u = x \log x$$

$[\because \log m^n = n \log m]$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = 1 + \log x$$

$$\Rightarrow \quad \frac{du}{dx} = u\,(1 + \log x)$$

$$\Rightarrow \quad \frac{du}{dx} = x^x\,(1 + \log x) \quad [\because u = x^x] \ldots\text{(ii)}$$ **(1)**

Also, $v = (\sin x)^x$

On taking log both sides, we get

$$\log v = \log (\sin x)^x$$

$$\Rightarrow \quad \log v = x \log (\sin x) \quad [\because \log m^n = n \log m]$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = x \cdot \frac{d}{dx}\log(\sin x) + \log(\sin x) \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + \log \sin x$$

$$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = x \cdot \frac{1}{\sin x} \cdot \cos x + \log \sin x$$

$$\left[\because \frac{d}{dx}(\log v) = \frac{1}{v}\frac{dv}{dx}\right]$$

$$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = x \cot x + \log \sin x$$

$$\Rightarrow \quad \frac{dv}{dx} = v(x \cot x + \log \sin x)$$

$$\Rightarrow \quad \frac{dv}{dx} = (\sin x)^x (x \cot x + \log \sin x) \quad \ldots\text{(iii)} \quad \textbf{(1)}$$

On putting the values of $\frac{du}{dx}$ from Eq. (ii) and $\frac{dv}{dx}$ from Eq. (iii) in Eq. (i), we get

$$\frac{dy}{dx} = x^x(1 + \log x) + (\sin x)^x (x \cot x + \log \sin x) \quad \textbf{(1)}$$

62. If $y = 3\cos(\log x) + 4\sin(\log x)$, then show that $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$. **Delhi 2009, 2009C**

To show, $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Given, $y = 3\cos(\log x) + 4\sin(\log x)$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = -3\sin(\log x)\frac{d}{dx}(\log x) + 4\cos(\log x)\frac{d}{dx}(\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

Again, differentiating both sides w.r.t. x , we get

$$x\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}(x)$$

$$= \frac{d}{dx}[-3\sin(\log x) + 4\cos(\log x)]$$ **(1½)**

$$\Rightarrow \quad x\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot 1=-3\cos(\log x)\frac{d}{dx}(\log x)$$
$$-4\sin(\log x)\frac{d}{dx}(\log x)$$

$$\Rightarrow x\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot 1=\frac{-3\cos(\log x)}{x}$$
$$-\frac{4\sin(\log x)}{x} \quad \textbf{(1½)}$$

$$\Rightarrow x\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{-[3\cos(\log x)+4\sin(\log x)]}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}$$
$$=-[3\cos(\log x)+4\sin(\log x)]$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y$$
$[\because 3\cos(\log x)+4\sin(\log x)=y$, given$]$

Hence, $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0 \quad \textbf{(1)}$$

Hence proved.

63. If $y=(x)^{\sin x}+(\log x)^x$, then find $\frac{dy}{dx}$. **Delhi 2009**

Do same as Que 61.

$$\left[\textbf{Ans. } \frac{dy}{dx}=x^{\sin x-1}[\sin x+x\log x\cos x] + (\log x)^{x-1}[1+\log x-\log(\log x)]\right]$$

64. If $x=a\left(\cos\theta+\log\tan\frac{\theta}{2}\right)$ and $y=a\sin\theta$,

then find the value of $\frac{d^2y}{dx^2}$ at $\theta=\frac{\pi}{4}$.

Delhi 2009C

Do same as Que 6.

$$\left[\textbf{Ans. } \left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{4}} = \frac{2\sqrt{2}}{a}\right]$$

65. If $y = (\log x)^x + (x)^{\cos x}$, then find $\frac{dy}{dx}$.

Delhi 2009C

Do same as Que 61.

$$\left[\begin{array}{l}\textbf{Ans. } (\log x)^{x-1}[1 + \log x \cdot \log(\log x)] \\ \qquad + x \cos^{x-1}[\cos x - x \log x \sin x]\end{array}\right]$$

66. If $y = e^x (\sin x + \cos x)$, then show that

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$

All India 2009

Given, $y = e^x (\sin x + \cos x)$ ...(i)

To show, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ ...(ii)

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\sin x + \cos x)$$

$$+ (\sin x + \cos x) \cdot \frac{d}{dx}(e^x)$$

$$= e^x (\cos x - \sin x) + (\sin x + \cos x) \cdot e^x$$

$$= e^x [\cos x - \sin x + \sin x + \cos x]$$

$$= e^x (2 \cos x)$$

$$\Rightarrow \quad \frac{dy}{dx} = 2e^x \cos x \quad \text{...(iii)} \quad \textbf{(1½)}$$

Again differentiating both sides of Eq. (iii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = (2e^x) \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(2e^x)$$

$$= 2e^x (-\sin x) + \cos x \cdot 2e^x$$

$$= 2e^x \cos x - 2e^x \sin x \quad \text{...(iv)} \quad \textbf{(1½)}$$

Now, we put the values of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ from Eqs. (iv) and (iii) along with value of y in LHS of Eq. (ii), we get

$$\text{LHS} = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y$$

$$= (2e^x \cos x - 2e^x \sin x) - 2(2e^x \cos x) + 2e^x (\sin x + \cos x)$$

$$= 2e^x \cos x - 2e^x \sin x - 4e^x \cos x + 2e^x \sin x + 2e^x \cos x$$

$$= 4e^x \cos x - 4e^x \cos x = 0 = \text{RHS} \quad \textbf{(1)}$$

Hence proved.

67. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, then show that

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0.$$

All India 2009

Given, $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2} \times \frac{d}{dx}(\sin^{-1} x) - (\sin^{-1} x) \times \frac{d}{dx}\sqrt{1-x^2}}{(\sqrt{1-x^2})^2}$$

$$= \frac{\left[\sqrt{1-x^2} \times \frac{1}{\sqrt{1-x^2}} - \sin^{-1} x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}(1-x^2)\right]}{(\sqrt{1-x^2})^2} \quad \textbf{(1)}$$

$$= \frac{\left[\sqrt{1-x^2} \cdot \frac{1}{\sqrt{1-x^2}} - (\sin^{-1} x) \cdot \frac{-2x}{2\sqrt{1-x^2}}\right]}{1-x^2}$$

$$= \frac{1 + \frac{x \sin^{-1} x}{\sqrt{1-x^2}}}{(1-x^2)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+xy}{1-x^2} \qquad \left[\because \frac{\sin^{-1} x}{\sqrt{1-x^2}} = y, \text{ given}\right]$$

$$\Rightarrow (1-x^2)\frac{dy}{dx} = 1 + xy \qquad \textbf{(1)}$$

Again differentiating above equation both sides w.r.t. x, we get

$$(1-x^2) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(1-x^2) = \frac{d}{dx}(1+xy)$$

(1)

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2x) = x\frac{dy}{dx} + y \cdot 1$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - x\frac{dy}{dx} - y = 0$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0 \qquad \textbf{(1)}$$

Hence proved.

68. Differentiate the following function w.r.t. x.

$(x)^{\cos x} + (\sin x)^{\tan x}$ **Delhi 2009**

Do same as Que 61.

$$\left[\textbf{Ans. } x^{\cos x - 1}[\cos x - x \log \sin x] + (\sin x)^{\tan x}[1 + \sec^2 x \log \sin x]\right]$$

69. Differentiate the following function w.r.t. x.

$x^{\sin x} + (\sin x)^{\cos x}$ **Delhi 2009**

Do same as Que 61.

$$\left[\textbf{Ans. } x^{\sin x - 1}[\sin x + x \log x \cos x] + \sin x^{\cos x}[\cos x \cot x - \sin x \log(\sin x)]\right]$$

70. If $y = x^{\cot x} + (\sin x)^x$, then find $\frac{dy}{dx}$.

All India 2008C

Do same as Que 61.

$$\left[\textbf{Ans. } x^{\cot x-1}[\cot x - x \log x \operatorname{cosec}^2 x] + (\sin x)^x [x \cot x + \log x \sin x]\right]$$

71. If $xy + y^2 = \tan x + y$, then find $\frac{dy}{dx}$.

All India 2008C

Firstly, differentiate the given expression with respect to x and then collect all the first derivative of y on one side to get the required result.

Given equation is $xy + y^2 = \tan x + y$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x + y) \quad \textbf{(1/2)}$$

$$\Rightarrow \left[x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(x)\right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

(1½)

$$\Rightarrow \quad x \cdot \frac{dy}{dx} + y \cdot 1 + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \quad \textbf{(1)}$$

$$\Rightarrow \quad (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1} \quad \textbf{(1)}$$

72. If $y = (\log x)^{\cos x} + \frac{x^2 + 1}{x^2 - 1}$, then find $\frac{dy}{dx}$.

Delhi 2008C

Do same as Que 37.

$$\left[\textbf{Ans. } (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x \log(\log x) - \frac{4x}{(x^2-1)^2}\right]\right]$$

73. If $y = \sin^{-1}\left[\frac{5x + 12\sqrt{1-x^2}}{13}\right]$, then find $\frac{dy}{dx}$.

HOTS; All India 2008

Do same as Que 56. $\left[\textbf{Ans. } \frac{1}{\sqrt{1-x^2}}\right]$

74. If $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$,

$0 < x < \frac{\pi}{2}$, then find $\frac{dy}{dx}$. HOTS; Delhi 2008

Firstly, convert the given function into simplest form by using

$$1+\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$$

and $1-\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}$

and then differentiate.

Given, $y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$ ...(i)

Now, on putting

$$1+\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}$$

and $1-\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}$

$-2\sin\frac{x}{2}\cos\frac{x}{2}$ in Eq. (i), we get **(1)**

$$y=\cot^{-1}\left[\frac{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}+\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}-\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$$

$$\Rightarrow y=\cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

(1)

$$\left[\begin{array}{l}\because a^2+b^2+2ab=(a+b)^2 \\ \text{and } a^2+b^2-2ab=(a-b)^2 \\ \text{and here, } a=\cos\frac{x}{2},\, b=\sin\frac{x}{2}\end{array}\right]$$

$$\Rightarrow y=\cot^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right]$$

$$\Rightarrow \quad y=\cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right] \qquad \textbf{(1)}$$

$$\Rightarrow \quad y=\cot^{-1}\cot\frac{x}{2} \qquad \left[\because \cot\theta=\frac{\cos\theta}{\sin\theta}\right]$$

$$\Rightarrow \quad y=\frac{x}{2} \qquad [\because \cot^{-1}\cot\theta=\theta]$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{2} \qquad \textbf{(1)}$$

NOTE (i) When $0<x<\frac{\pi}{2}$, then we consider

$$\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}$$

(ii) When $\frac{\pi}{2}<x<\pi$, then we consider

$$\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\sqrt{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}$$

75. If $y = \sqrt{x^2+1} - \log\left(\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right)$, then find $\frac{dy}{dx}$.

Delhi 2008

Given, $y = \sqrt{x^2+1} - \log\left(\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right)$

It can be written as

$$y = \sqrt{x^2+1} - \log\left(\frac{1}{x} + \sqrt{\frac{x^2+1}{x^2}}\right)$$

$$\Rightarrow \quad y = \sqrt{x^2+1} - \log\left(\frac{\sqrt{x^2+1}+1}{x}\right) \quad \textbf{(1/2)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x^2+1}\right) - \frac{d}{dx}\log\left(\frac{\sqrt{x^2+1}+1}{x}\right)$$

$$= \frac{1}{2\sqrt{x^2+1}}\frac{d}{dx}(x^2+1) - \frac{d}{dx}\log\left(\frac{\sqrt{x^2+1}+1}{x}\right)$$

$$= \frac{2x}{2\sqrt{x^2+1}} - \frac{1}{\left(\frac{\sqrt{x^2+1}+1}{x}\right)} \times \frac{d}{dx}\left(\frac{\sqrt{x^2+1}+1}{x}\right) \quad \textbf{(1)}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}+1} \times \frac{x \cdot \frac{d}{dx}(\sqrt{x^2+1}+1) - (\sqrt{x^2+1}+1)\frac{d}{dx}(x)}{x^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\quad dx \qquad dx}{v^2}\right]$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}+1} \times \frac{x\left(\dfrac{2x}{2\sqrt{x^2+1}}\right) - (\sqrt{x^2+1}+1)\cdot 1}{x^2}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}+1} \times \frac{\dfrac{x^2}{\sqrt{x^2+1}} - (\sqrt{x^2+1}+1)}{x^2}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}+1}$$

$$\times \frac{x^2 - (\sqrt{x^2+1})(\sqrt{x^2+1}+1)}{x^2(\sqrt{x^2+1})}$$

$$= \frac{x}{\sqrt{x^2+1}} - \left[\frac{x^2 - x^2 - 1 - \sqrt{x^2+1}}{x(\sqrt{x^2+1}+1)(\sqrt{x^2+1})}\right]$$

(1½)

$$= \frac{x}{\sqrt{x^2+1}} - \frac{(-\sqrt{x^2+1}-1)}{x(\sqrt{x^2+1}+1)(\sqrt{x^2+1})}$$

$$= \frac{1}{\sqrt{x^2+1}}\left[x + \frac{(\sqrt{x^2+1}+1)}{x(\sqrt{x^2+1}+1)}\right]$$

$$= \frac{1}{\sqrt{x^2+1}}\left(x + \frac{1}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} \cdot \frac{(x^2+1)}{x}$$

Hence, $\frac{dy}{dx} = \frac{\sqrt{x^2+1}}{x}$ **(1)**

76. Differentiate the following function w.r.t. x.

$$\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

Delhi 2008

> Reduce the given function into simplest form by putting $x = \cos\theta$ and by using the half angle formulae
>
> $$1 + \cos\theta = 2\cos^2\frac{\theta}{2} \text{ and } 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$
>
> Then, find its derivative with respect to x.

Given function is

$$y = \tan^{-1}\left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right]$$

On putting $x = \cos\theta$

$\Rightarrow \quad \theta = \cos^{-1} x$, we get

$$\therefore \quad y = \tan^{-1}\left[\frac{\sqrt{1+\cos\theta} - \sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}}\right] \quad \ldots\text{(i)} \ \mathbf{(1)}$$

We know that,

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2}$$

and

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

On putting the above values in Eq. (i), we get

$$y = \tan^{-1}\left[\frac{\sqrt{2\cos^2\frac{\theta}{2}} - \sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\cos^2\frac{\theta}{2}} + \sqrt{2\sin^2\frac{\theta}{2}}}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2} - \sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2} + \sqrt{2}\sin\frac{\theta}{2}}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cdots}\right) \quad \mathbf{(1)}$$

$$\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)$$

On dividing numerator and denominator by $\cos\frac{\theta}{2}$, we get

$$y=\tan^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right)$$

$$\Rightarrow \quad y=\tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right)$$

$$\left[\because \quad 1=\tan\frac{\pi}{4} \quad \text{and } \tan\frac{\theta}{2}=1\times\tan\frac{\theta}{2}=\tan\frac{\pi}{4}\tan\frac{\theta}{2}\right]$$

$$\Rightarrow \quad y=\tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\left[\because \frac{\tan A-\tan B}{1+\tan A\tan B}=\tan(A-B)\right]$$

$$\Rightarrow \quad y=\frac{\pi}{4}-\frac{\theta}{2} \qquad [\because \tan^{-1}\tan x=x]$$

$$\Rightarrow \quad y=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x \qquad [\because \theta=\cos^{-1}x]$$ **(1)**

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=0-\frac{1}{2}\times\frac{-1}{\sqrt{1-x^2}}\left[\because \frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}\right]$$

Hence, $\frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}$ **(1)**